Signal Processing and Speech Communication Laboratory—Prof. G. Kubin
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Exam *Adaptive Systems* on 2015/2/2

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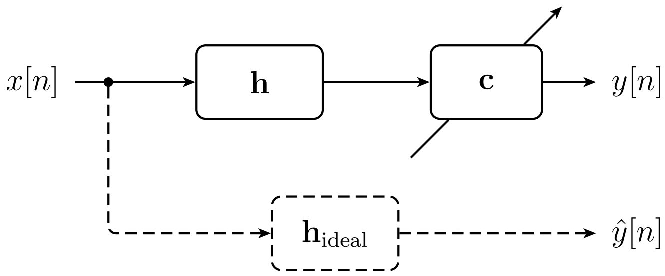
Exam duration: 3 hours
Maximum points: 100
Allowed material: mathematical formulary, calculator
You have to return this document after the exam.
Best wishes for a successful exam!

Problem 1 (33 Points)



We are given the impulse response vector $\mathbf{h} = [1,\ 0.5]^T$ of an LTI system. We further denote with $\mathbf{H}$ the *convolution matrix*, whose columns are composed of shifted vectors $\mathbf{h}$, i.e.,

$$\mathbf{H} = \begin{bmatrix} 1 & 0 & 0 & \cdots \\ 0.5 & 1 & 0 & \cdots \\ 0 & 0.5 & 1 & \cdots \\ 0 & 0 & 0.5 & \ddots \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix}.$$

We now want to design an FIR filter with coefficient vector $\mathbf{c}$ such that the cascade of $\mathbf{c}$ and $\mathbf{h}$ approximates the desired impulse response vector $\mathbf{h}_{\text{ideal}}$ optimally in the LS sense.

(a) Using the error vector $\mathbf{e} = \mathbf{H}\mathbf{c} - \mathbf{h}_{\text{ideal}}$, show that the optimal solution is given by

$$\mathbf{c}_{LS} = (\mathbf{H}^T\mathbf{H})^{-1}\mathbf{H}^T\mathbf{h}_{\text{ideal}}.$$

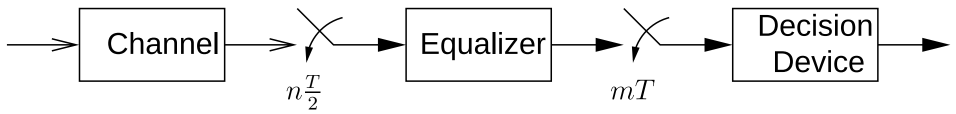
Use a general dimension $N = \dim(\mathbf{c})$ of the coefficient vector.

(b) Let $\mathbf{h}_{\text{ideal}}$ be such that $\hat{y}[n] = x[n]$ (i.e., we design a delay-free equalizer). Compute the two-dimensional ($N = 2$) coefficient vector $\mathbf{c}$ optimal in the LS sense. Make sure that you use the correct dimensions for $\mathbf{h}_{\text{ideal}}$ and $\mathbf{H}$!

(c) Repeat task **(b)** for a three-dimensional ($N = 3$) coefficient vector[1].

(d) Plot the pole-/zero diagram of the z-transform of the LTI system with coefficient vector $\mathbf{h}$. Can you invert this system? If yes, how could you accomplish this? What is the difference between the LS solution and the solution obtained by directly inverting the system?

Problem 2 (34 Points)



Consider the $\frac{T}{2}$-*fractionally-spaced equalizer* illustrated above where the incoming signal is sampled at a rate twice the symbol rate. The decision device is synchronized with the even-indexed samples. The discrete-time FIR description of the communication channel for the high sampling rate is

$$H(z) = h_0 + h_1 z^{-1} + h_2 z^{-2} + h_3 z^{-3} = 1/2 + 1\,z^{-1} + 1/2\,z^{-2} + 1/4\,z^{-3}.$$

(Note, the unit delay z^{-1} corresponds to $\frac{T}{2}$ here.)

(a) Assume the transmitted symbols to be ± 1 and consider the receiver without the equalizer (replace it by a straight line). For the given channel, can the open-eye condition be met? Hint: which samples of the channel's impulse response influence the transmitted, T-spaced symbols?

(b) Calculate the 3 coefficients of the $\frac{T}{2}$-fractionally-spaced equalizer

$$C(z) = c_0 + c_1 z^{-1} + c_2 z^{-2}$$

such that the cascade of the given channel and the equalizer is (or approximates) $H(z)C(z) = 1$, i.e., enables a delay-free and ISI-free transmission.

(c) Calculate the 3 coefficients of the $\frac{T}{2}$-fractionally-spaced equalizer such that the cascade is (or approximates) a delay of 1 symbol.

(d) Calculate the 3 coefficients of the $\frac{T}{2}$-fractionally-spaced equalizer such that the cascade is (or approximates) a delay of 2 symbols.

(e) Consider the channel to be noisy. Compute the noise gains of the three equalizers of the previous tasks. Which one of the three equalizers should be chosen?

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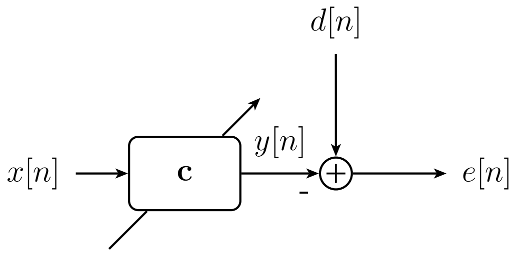
[1]The inverse of a symmetric 3×3 Toeplitz-matrix is given as

$$\begin{bmatrix} a & b & c \\ b & a & b \\ c & b & a \end{bmatrix}^{-1} = \frac{1}{a^3 - 2ab^2 + 2b^2c - ac^2} \begin{bmatrix} a^2 - b^2 & bc - ba & b^2 - ac \\ bc - ba & a^2 - c^2 & bc - ba \\ b^2 - ac & bc - ba & a^2 - b^2 \end{bmatrix}.$$

Problem 3 (33 Points)



We want to design a filter according to some specifications with the help of the Wiener-Hopf equation. To this end, we assume that the input signal $x[n]$ is given as

$$x[n] = \cos\left(\frac{\pi}{3}n + \varphi_1\right) + \cos\left(\frac{2\pi}{3}n + \varphi_2\right)$$

where φ_1 and φ_2 are independent random variables, both uniformly distributed on $[-\pi, \pi)$. The autocorrelation sequence of this input signal thus computes to

$$r_{xx}[k] = \mathbb{E}\left(x[n]x[n-k]\right) = \frac{1}{2}\cos\left(\frac{\pi}{3}k\right) + \frac{1}{2}\cos\left(\frac{2\pi}{3}k\right)$$

The desired output of the filter is given as

$$d[n] = \cos\left(\frac{\pi}{3}(n-1) + \varphi_1\right).$$

You can interpret $d[n]$ as the output of an ideal low-pass filter with a cut-off frequency between $\frac{\pi}{3}$ and $\frac{2\pi}{3}$ and a group delay of one sample.

(a) Using this input signal, what is the maximum order $\dim(\mathbf{c}) = N$ you can use for the adaptive filter? **Hint:** Think about *persistence of excitation*!

(b) Calculate the cross-correlation vector $\mathbf{p} = \mathbb{E}\left(d[n]\mathbf{x}[n]\right)$.

(c) Compute the optimal filter coefficients for a first-order filter, i.e., compute $\mathbf{c}_{\text{opt}}$ for $N = 2$. Use the Wiener-Hopf equation.

(d) Compute[2] the optimal filter coefficients for a second-order filter, i.e., compute $\mathbf{c}_{\text{opt}}$ for $N = 3$.

(e) For the result of **(d)** compute the transfer function $C(z)$ and its zeros. Then, evaluate the magnitude response $|C(e^{j\theta})|$ for $\theta = \frac{\pi}{3}$ and for $\theta = \frac{2\pi}{3}$. What do you observe?

[2]The inverse of a symmetric 3×3 Toeplitz-matrix is given as

$$\begin{bmatrix} a & b & c \\ b & a & b \\ c & b & a \end{bmatrix}^{-1} = \frac{1}{a^3 - 2ab^2 + 2b^2c - ac^2}\begin{bmatrix} a^2-b^2 & bc-ba & b^2-ac \\ bc-ba & a^2-c^2 & bc-ba \\ b^2-ac & bc-ba & a^2-b^2 \end{bmatrix}.$$