Signal Processing and Speech Communication Laboratory—Dr. C. Steiner
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Exam *Adaptive Systems* on 2014/7/11

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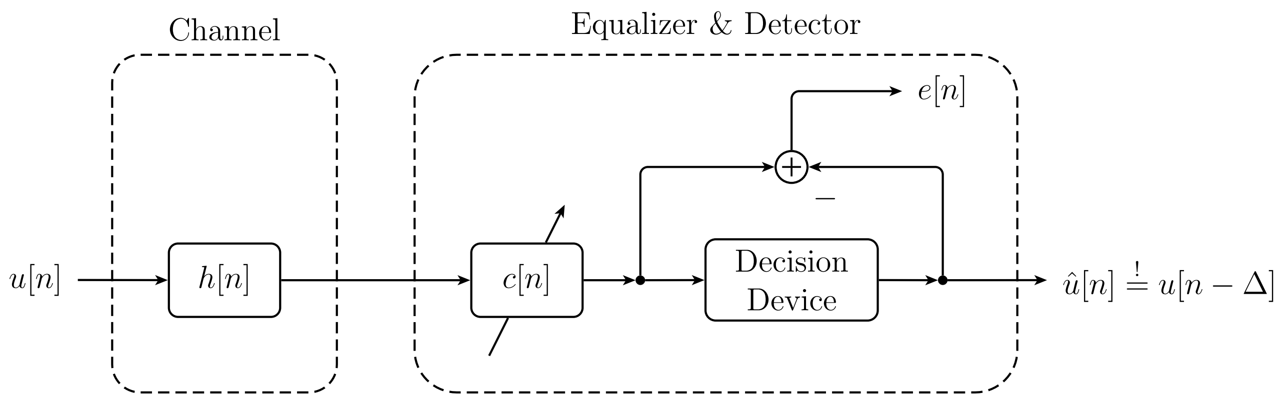
Exam duration: 3 hours
Maximum points: 100
Allowed material: mathematical formulary, calculator
You have to return this document after the exam. Best wishes for a successful exam!

Problem 1 (33 Points)

Consider the given data transmission scenario over a channel with impulse response $h[n]$. We use an equalizer in decision-directed operation mode.



The symbols to be transmitted are $u[n] \in \{-1, 1\}$, which occur with the same probability. Consecutive symbols can be assumed to be uncorrelated. The decision device returns the sign of its input signal (**zero is regarded as a positive number**). The channel is given as

$$h[n] = \delta[n] + \delta[n-1].$$

(a) Show that $\mathbf{R}_{uu} = \mathbf{I}$.

(b) Consider a transmission over the given channel *without* an equalizer (decision device only). Is there an ideal delay Δ that minimizes ISI (Inter Symbol Interference)? Calculate the worst-case ISI and answer whether the channel's eye is open or not.

(c) Determine the sequence(s) for which the worst-case ISI occurs.

(d) Show that for an arbitrary choice of N and Δ the optimum solution $\mathbf{c}_{\text{opt}}$ in the sense of a minimum mean-squared error (MinMSE) is the $(\Delta+1)$-th column of the pseudo-inverse of the convolution matrix of $\mathbf{h}$, i.e., of

$$\left(\mathbf{H}^T\mathbf{H}\right)^{-1}\mathbf{H}^T$$

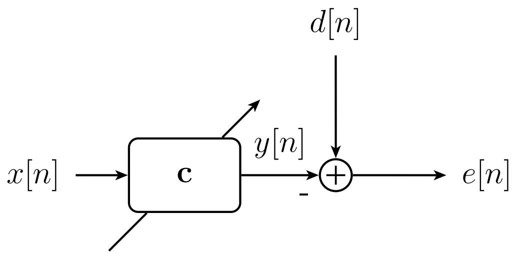
Make sure that you write down the matrix/vector dimensions explicitly!

(e) Now choose $N = 2$ and $\Delta = 0$ and determine the optimum filter coefficient vector $\mathbf{c}_{\text{opt}}$. Calculate the noise gain of the equalizer. Can this equalizer open the channel's eye? If not, write down again the input sequence(s) for which an error occurs.

(f) Repeat task **(e)** for $\Delta = 1$ and compare the results to those of **(e)**. Which delay Δ is preferable? Justify your answer!

Problem 2 (34 Points)

Consider the following linear filtering problem:



The auto-correlation sequence of $x[n]$ and the cross-correlations between $x[n-k]$ and $d[n]$ are assumed to be known (we can build the auto-correlation matrix $\mathbf{R}_{xx}$ and the cross-correlation vector $\mathbf{p}$). The following adaptation rule (*coefficient-leakage gradient search*) is used to adapt the coefficient vector $\mathbf{c}[n]$:

$$\mathbf{c}[n] = (1-\mu\alpha)\mathbf{c}[n-1] + \mu(\mathbf{p} - \mathbf{R}_{xx}\mathbf{c}[n-1])$$

where α is the leakage parameter $(0 < \alpha \ll 1)$ and μ is the step-size parameter.

(a) Assume convergence. Where does this algorithm converge to?

(b) Transform the given adaptation rule in a way such that it adapts the misalignment vector (defined as $\mathbf{v}[n] = \mathbf{c}[n] - \mathbf{c}_\infty$). Hint: substitute for $\mathbf{p}$ using the result from **(a)**.

(c) Decouple the adaptation rule of **(b)** into a set of scalar adaptation expressions by using a unitary transform $\mathbf{q}[n] = \mathbf{Q}^H\mathbf{v}[n]$.

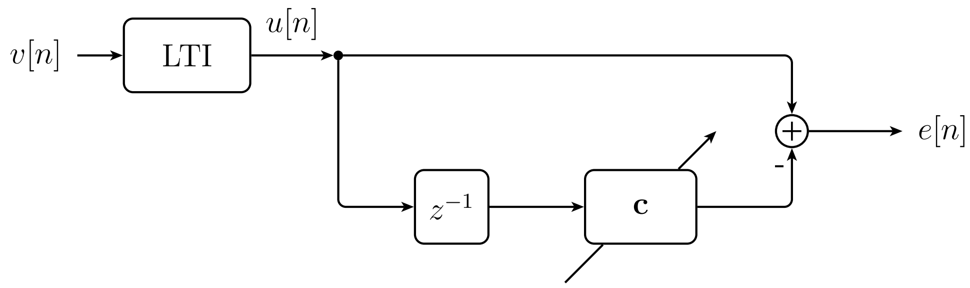
(d) Write the decoupled equation from **(c)** as individual exponential sequences and derive a condition on μ to ensure convergence towards $\mathbf{c}_\infty$, i.e., specify the range $\mu_{min} < \mu < \mu_{max}$ (assume α to be given).

(e) Assume we use a fixed step-size parameter μ. Compute the worst-case convergence time constant τ_{max}.

(f) Now set $\mathbf{R}_{xx} = \mathbf{I}$, $\mu = 0.01$, and $\alpha = 0$. How many iterations are necessary until the largest component of the misalignment vector $\mathbf{v}[n]$ reduced to only 5% of its initial value?

Problem 3 (33 Points)

A prediction-error filter can be considered as a blind equalizer for AR processes.



The first 3 samples of the auto-correlation sequence of $u[n]$ are given as $r_{uu}[0] = 1.0$, $r_{uu}[1] = -0.\bar{5} = -\frac{5}{9}$, and $r_{uu}[2] = \frac{25}{81} \approx 0.309$ where the bar denotes repeating decimals.

(a) Derive the MSE-optimal coefficient vector $\mathbf{c}_{\text{opt}}$ for a general predictor order N. **Hint:** You may use the Wiener-Hopf solution, but make sure that you adapt it properly to the signal model relevant in this example.

(b) For a predictor with $N = 1$ coefficient, find the MSE-optimal coefficient of the adaptive filter.

(c) For a predictor with $N = 2$ coefficients, find the MSE-optimal coefficient vector $\mathbf{c}$ of the adaptive filter.

(d) Given your state of information, what is the order of the AR process? Write down the process-generator difference equation and calculate the variance σ_v^2 of the white-noise input $v[n]$.

(e) Next, assume that the coefficient vector $\mathbf{c}$ ($N = 2$) is updated constantly, such that it is MSE-optimal. Assume now that the input $v[n]$ is suddenly switched from a white to a non-white signal. How would the coefficients change?