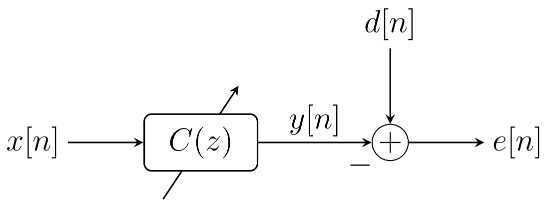
Signal Processing and Speech Communication Laboratory—Prof. G. Kubin
Graz University of Technology

Exam *Adaptive Systems* on 2019/3/22

Name MatrNr. StudKennz.

Exam duration: 3 hours
Maximum points: 100
Allowed material: mathematical formulary, calculator
You have to return this document after the exam.
Best wishes for a successful exam!

Problem 1 (34 Points)



We want to design a filter according to some specifications with the help of the Wiener-Hopf equation. To this end, we assume that the input signal $x[n]$ is given as

$$x[n] = \cos\left(\frac{\pi}{3}n + \varphi_1\right) + \cos\left(\frac{2\pi}{3}n + \varphi_2\right)$$

where φ_1 and φ_2 are independent random variables, both uniformly distributed on $[-\pi, \pi)$. The autocorrelation sequence of this input signal thus computes to

$$r_{xx}[k] = \mathbb{E}\left(x[n]x[n-k]\right) = \frac{1}{2}\cos\left(\frac{\pi}{3}k\right) + \frac{1}{2}\cos\left(\frac{2\pi}{3}k\right)$$

The desired output of the filter is given as

$$d[n] = \cos\left(\frac{\pi}{3}(n-1) + \varphi_1\right).$$

You can interpret $d[n]$ as the output of an ideal low-pass filter with a cut-off frequency between $\frac{\pi}{3}$ and $\frac{2\pi}{3}$.

(a) Using this input signal, what is the maximum order $\dim(\mathbf{c}) = N$ you can use for the adaptive filter? **Hint:** Think about *persistence of excitation*!

(b) Calculate the cross-correlation vector $\mathbf{p} = \mathbb{E}\left(d[n]\mathbf{x}[n]\right)$.

(c) Compute the optimal filter coefficients for a first-order filter, i.e., compute $\mathbf{c}_{\text{opt}}$ for $N = 2$. Use the Wiener-Hopf equation.

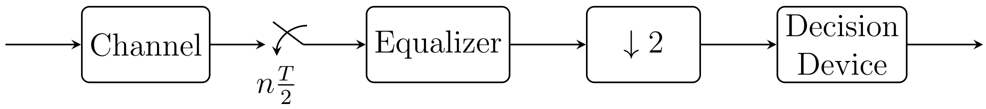
(d) Compute[1] the optimal filter coefficients for a second-order filter, i.e., compute $\mathbf{c}_{\text{opt}}$ for $N = 3$.

(e) For the result of **(d)** compute the transfer function $C(z)$ and its zeros. Then, evaluate the magnitude response $|C(e^{j\theta})|$ for $\theta = \frac{\pi}{3}$ and for $\theta = \frac{2\pi}{3}$. What do you observe?

[1] The inverse of a symmetric 3×3 Toeplitz-matrix is given as

$$\begin{bmatrix} a & b & c \\ b & a & b \\ c & b & a \end{bmatrix}^{-1} = \frac{1}{a^3 - 2ab^2 + 2b^2c - ac^2} \begin{bmatrix} a^2 - b^2 & bc - ba & b^2 - ac \\ bc - ba & a^2 - c^2 & bc - ba \\ b^2 - ac & bc - ba & a^2 - b^2 \end{bmatrix}.$$

Problem 2 (34 Points)



Consider the $\frac{T}{2}$-*fractionally-spaced equalizer* illustrated above where the incoming signal is sampled at a rate twice the symbol rate T. The decision device is synchronized with the even-indexed samples. The discrete-time FIR description of the communication channel for the high sampling rate is

$$H(z) = h_0 + h_1 z^{-1} + h_2 z^{-2} + h_3 z^{-3} = 1/2 + 1\, z^{-1} + 1/2\, z^{-2} + 1/4\, z^{-3}.$$

(Note, the unit delay z^{-1} corresponds to $\frac{T}{2}$ here.)

(a) Assume the transmitted symbols to be ± 1 and consider the receiver without the equalizer (replace it by a straight line). For the given channel, can the open-eye condition be met?
Hint: Which samples of the channel's impulse response influence the transmitted, T-spaced symbols?

(b) Calculate the 3 coefficients of the $\frac{T}{2}$-fractionally-spaced equalizer

$$C(z) = c_0 + c_1 z^{-1} + c_2 z^{-2}$$

such that the cascade of the given channel and the equalizer enables a delay-free and ISI-free transmission.
Hint: Again not all samples have to be considered.

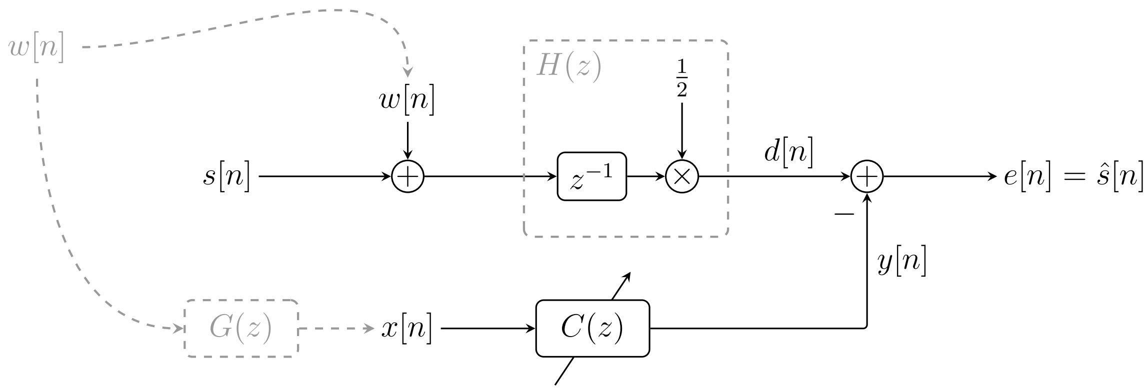
(c) Calculate the 3 coefficients of the $\frac{T}{2}$-fractionally-spaced equalizer such that the cascade approximates a delay of 1 symbol.

(d) Calculate the 3 coefficients of the $\frac{T}{2}$-fractionally-spaced equalizer such that the cascade approximates a delay of 2 symbols.

(e) Consider the channel to be noisy. Compute the noise gains of the three equalizers of the previous tasks. Which one of the three equalizers should be chosen?

Problem 3 (32 Points)

Consider the adaptive noise-cancelation problem shown below:



A speech signal $s[n]$ (for simplicity, it is assumed that $s[n]$ is a white Gaussian noise signal with unit variance) is superimposed with statistically independent white Gaussian noise $w[n]$ with some variance σ_w^2. Unfortunately the noise reference is filtered by an IIR-filter $G(z)$ before it is captured. The filter is specified by the purely recursive difference equation

$$x[n] = w[n] + \frac{1}{2}x[n-1].$$

You should now design an adaptive filter which has access to $x[n]$ in order to reconstruct $s[n]$ as accurately as possible.

(a) Compute σ_w^2 and the autocorrelation sequence $r_{xx}[k]$ of $x[n]$, assuming that $x[n]$ has unit variance.

(b) For $N = 3$, determine the autocorrelation matrix $\mathbf{R}_{xx} = \mathbb{E}\{\mathbf{x}[n]\mathbf{x}^\mathsf{T}[n]\}$ of the signal $x[n]$ and the crosscorrelation vector $\mathbf{p} = \mathbb{E}\{d[n]\mathbf{x}[n]\}$.

(c) Determine the Wiener-Hopf solution $\mathbf{c}_{\text{opt}}$ for the second-order filter $\mathbf{c}$ with $N = 3$ coefficients.[2]

(d) Is the adaptive filter able to identify $H(z)$? Is it able remove the interfering noise $w[n]$, even though $G(z)$ is an IIR-filter? Explicitly write down $y[n]$ for your $\mathbf{c}_{\text{opt}}$ obtained in **(c)**.

(e) Can you identify the system $G(z)$ when you assume that you have perfect knowledge of the system $H(z)$ and the Wiener-Hopf solution $\mathbf{c}_{\text{opt}}$? If yes, show how this would work for the given systems, if not explain why this is not possible.

[2]The inverse of a symmetric 3×3 Toeplitz-matrix is given as

$$\begin{bmatrix} a & b & c \\ b & a & b \\ c & b & a \end{bmatrix}^{-1} = \frac{1}{a^3 - 2ab^2 + 2b^2c - ac^2} \begin{bmatrix} a^2 - b^2 & bc - ba & b^2 - ac \\ bc - ba & a^2 - c^2 & bc - ba \\ b^2 - ac & bc - ba & a^2 - b^2 \end{bmatrix}.$$