Signal Processing and Speech Communication Laboratory—Dr. C. Steiner
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Exam *Adaptive Systems* on 2013/9/23

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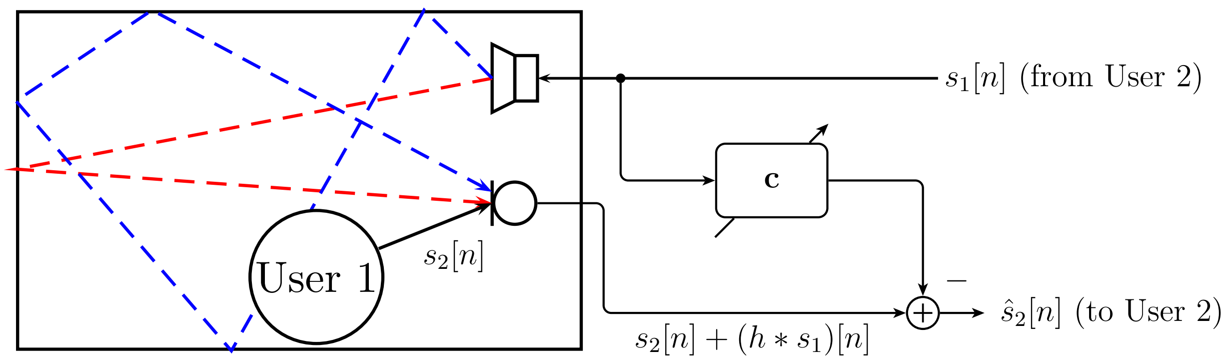
Exam duration: 3 hours
Maximum points: 100
Allowed material: mathematical formulary, calculator
You have to return this document after the exam. Best wishes for a successful exam!

Problem 1 (34 Points)

Imagine you are working at a speech technology company, and it is your task to design a voice-over-the-internet communication system (let's call it *Sqype*). The first problem you encounter is the following: Assume your users use a standard PC with an external speaker system and a hand-held microphone. It may occur that the signal from the speakers is captured by the microphone and transmitted to user on the other end of the line: the other user will hear herself with a slight delay, a very annoying phenomenon called *echo*. It is your task to design a digital filter which compensates that echo. In other words, you want to achieve $\hat{s}_2[n] = s_2[n]$.



We restrict ourselves to a first-order ($N = 2$) adaptive filter to keep the end-to-end delay small. Furthermore, we assume that the impulse response $h[n]$ (combining the effects of the microphone, the speaker, and the multipath propagation in the room) is given as

$$h[n] = \frac{13}{12}\delta[n] + \frac{5}{6}\delta[n-1] + \frac{1}{4}\delta[n-2].$$

(a) Assume that $s_1[n]$ and $s_2[n]$ are independent, zero-mean, white noise processes with variances σ_1^2 and σ_2^2. Derive the filter coefficients $\mathbf{c}_{\text{opt}}$ optimal in the MSE-sense, i.e., minimize

$$E\left\{(s_2[n] - \hat{s}_2[n])^2\right\}.$$

(b) Under the assumption made above, compute the mean-squared error energy.
We are now looking at a more natural scenario: The signals $s_1[n]$ and $s_2[n]$ are still independent and zero-mean, but like speech signals we can assume that they have a low-pass character. In other words, for $i = 1, 2$,

$$s_i[n] = \frac{1}{\sqrt{2}}\left(w_i[n] + w_i[n-1]\right).$$

where $w_i[n]$ are now independent, white noise processes with variances σ_1^2 and σ_2^2.

(c) Verify that the variance of $s_i[n]$ is the same as the variance of $w_i[n]$.

(d) Compute the autocorrelation matrix $\mathbf{R}_{s_1 s_1}$.

(e) Derive again the filter coefficients $\mathbf{c}_{\text{opt}}$ optimal in the MSE-sense. What has changed compared to the case where $s_1[n]$ was white?

(f) Does the fact that $s_2[n]$ has a low-pass character influence the results in any way?

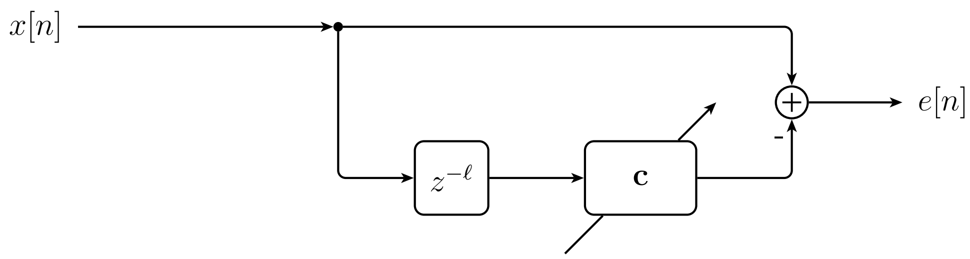
(g) Assuming finally that you compute the optimal filter coefficients iteratively, i.e., using either an LMS or a gradient search algorithm, under which signal conditions will the coefficients converge faster? In the white noise or in the low-pass case? Justify your answer!

Problem 2 (34 Points)

We consider a simple linear prediction filter, which is optimized in the MSE-sense. In other words, $\mathbf{c}$ is chosen such that prediction error

$$\text{MSE} = E\left\{(x[n] - \mathbf{c}^T \mathbf{x}[n-\ell])^2\right\}$$

is minimized.



(a) Show that the coefficient vector of the MSE-optimal N-coefficient, lag-ℓ prediction filter is given as

$$\mathbf{c}_{\text{opt}} = \mathbf{R}_{xx}^{-1} \begin{bmatrix} r_{xx}[\ell] \\ r_{xx}[\ell+1] \\ r_{xx}[\ell+2] \\ \vdots \\ r_{xx}[\ell+N-1] \end{bmatrix}$$

where r_{xx} and $\mathbf{R}_{xx}$ are the autocorrelation function and the autocorrelation matrix of $x[n]$, respectively.

(b) Derive the prediction error as a function of $\mathbf{c}$, r_{xx}, and $\mathbf{R}_{xx}$.

Now we assume that the values of the autocorrelation function $r_{xx}[m]$ for $m = 0, 1, 2, \ldots$ are given as

$$\{2, -1, 1, -1, 1, -1, \ldots\}.$$

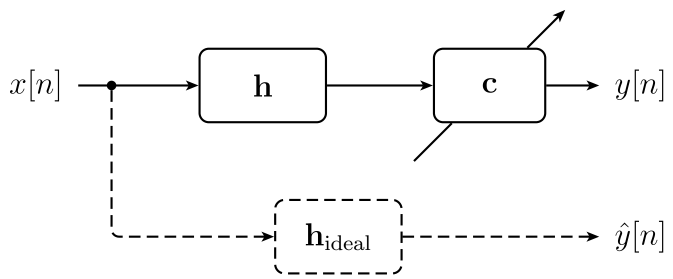
(c) What kind of process could give rise to such an autocorrelation function?

(d) Compute the coefficients of the optimal lag-1 prediction filter for $N = 1$ and $N = 2$. In addition to that, compute the prediction error for both cases.

(e) Repeat the previous task for a lag-2 prediction filter. What can you observe? In particular, compare the prediction error of to the values obtained in **(d)**.

(f) One would expect that a larger lag leads to a larger prediction error. Why is this not the case in this example?

Problem 3 (32 Points)



We are given the impulse response vector $\mathbf{h} = [1,\ 0.5]^T$ of an LTI system. We further denote with $\mathbf{H}$ the *convolution matrix*, whose columns are composed of shifted vectors $\mathbf{h}$, i.e.,

$$\mathbf{H} = \begin{bmatrix} 1 & 0 & 0 & \cdots \\ 0.5 & 1 & 0 & \cdots \\ 0 & 0.5 & 1 & \cdots \\ 0 & 0 & 0.5 & \ddots \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix}.$$

We now want to design an FIR filter with coefficient vector $\mathbf{c}$ such that the cascade of $\mathbf{c}$ and $\mathbf{h}$ approximates the desired impulse response vector $\mathbf{h}_{\text{ideal}}$ optimally in the LS sense.

(a) Using the error vector $\mathbf{e} = \mathbf{Hc} - \mathbf{h}_{\text{ideal}}$, show that the optimal solution is given by

$$\mathbf{c}_{LS} = (\mathbf{H}^T\mathbf{H})^{-1}\mathbf{H}^T\mathbf{h}_{\text{ideal}}.$$

Use a general dimension $N = \dim(\mathbf{c})$ of the coefficient vector.

(b) Let $\mathbf{h}_{\text{ideal}}$ be such that $\hat{y}[n] = x[n]$ (i.e., we design a delay-free equalizer). Compute the two-dimensional ($N = 2$) coefficient vector $\mathbf{c}$ optimal in the LS sense. Make sure that you use the correct dimensions for $\mathbf{h}_{\text{ideal}}$ and $\mathbf{H}$!

(c) Repeat task **(b)** for a three-dimensional ($N = 3$) coefficient vector[1].

(d) Plot the pole-/zero diagram of the z-transform of the LTI system with coefficient vector $\mathbf{h}$. Can you invert this system? If yes, how could you accomplish this? What is the difference between the LS solution and the solution obtained by directly inverting the system?

[1] The inverse of a symmetric 3×3 Toeplitz-matrix is given as

$$\begin{bmatrix} a & b & c \\ b & a & b \\ c & b & a \end{bmatrix}^{-1} = \frac{1}{a^3 - 2ab^2 + 2b^2c - ac^2} \begin{bmatrix} a^2 - b^2 & bc - ba & b^2 - ac \\ bc - ba & a^2 - c^2 & bc - ba \\ b^2 - ac & bc - ba & a^2 - b^2 \end{bmatrix}.$$