Signal Processing and Speech Communication Laboratory—Prof. G. Kubin
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Exam *Adaptive Systems* on 2018/6/29

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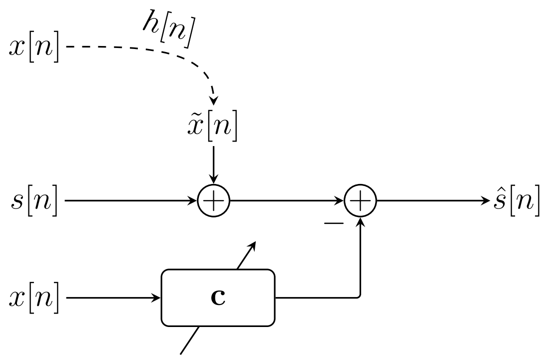
Exam duration: 3 hours
Maximum points: 100
Allowed material: mathematical formulary, calculator
You have to return this document after the exam.
Best wishes for a successful exam!

Problem 1 (33 Points)

Consider the following interference-cancelation problem:



A signal $s[n]$ (for simplicity, it is assumed that $s[n]$ is a white-noise signal with unit variance) is superimposed with a periodic interference $\tilde{x}[n]$, which is itself a filtered version of a periodic signal

$$x[n] = \cos\left(n\frac{\pi}{2} + \varphi\right)$$

where φ is uniformly distributed in $[0, 2\pi)$. It follows that the autocorrelation function of $x[n]$ is given as

$$r_{xx}[k] = \frac{1}{2}\cos\left(k\frac{\pi}{2}\right).$$

Assume further that the filter between $x[n]$ and $\tilde{x}[n]$ is a first-order IIR system with impulse response

$$h[n] = \left(\frac{1}{\sqrt{3}}\right)^n u[n].$$

You should now design an adaptive filter which has access to $x[n]$ in order to reconstruct $s[n]$ as accurately as possible, i.e., you should minimize

$$\mathbb{E}\{e^2[n]\} = \mathbb{E}\{(\hat{s}[n] - s[n])^2\}.$$

(a) What is the maximum filter length N you can choose for the adaptive (finite-impulse-response) filter **c**? Justify your answer!

(b) Note that the signal $\tilde{x}[n]$ can be written as

$$\tilde{x}[n] = A\cos\left(n\frac{\pi}{2} + \varphi + \phi\right)$$

where A and ϕ are amplitude and phase changes introduced by the filter $h[n]$. Compute A and ϕ for the impulse response given above[1].

(c) For $N = 2$, determine the autocorrelation matrix $\mathbf{R}_{xx} = \mathbb{E}\{\mathbf{x}[n]\mathbf{x}[n]^T\}$ of the signal $x[n]$ and the crosscorrelation vector $\mathbf{p} = \mathbb{E}\{\tilde{x}[n]\mathbf{x}[n]\}$.

[1] If you cannot do it, you can use A and ϕ (instead of their values) in all following tasks.

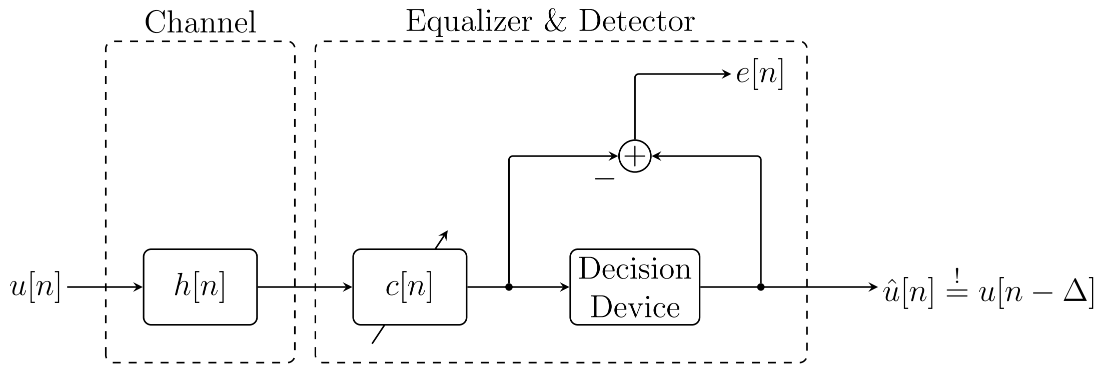
(d) Determine the Wiener solution $\mathbf{c}_{\text{opt}}$ for the first-order filter $\mathbf{c}$ ($N = 2$).

(e) Show that for the Wiener solution the mean-squared error vanishes, i.e., that $\mathbb{E}\{e^2[n]\} = 0$.

(f) As you can see, the Wiener solution yields the optimal result. What if we try to cancel the interference with an adaptive filter adapted by the LMS algorithm? Would $\mathbb{E}\{e^2[n]\} = 0$ still hold for the LMS (i.e. in the asymptotic case as $n \to \infty$)?

Problem 2 (34 Points)

Consider the given data transmission scenario over a channel with impulse response $h[n]$. We use an equalizer in decision-directed operation mode.



The symbols to be transmitted are $u[n] \in \{-1, 1\}$, which occur with the same probability. Consecutive symbols can be assumed to be uncorrelated. The decision device returns the sign of its input signal (**zero is regarded as a positive number**). The channel is given as

$$h[n] = \delta[n] + \delta[n-1].$$

(a) What assumption do we implicitly make when using an equalizer in decision-directed mode?

(b) Consider a transmission over the given channel *without* an equalizer (decision device only). Is there an ideal delay Δ that minimizes ISI (Inter Symbol Interference)? Calculate the worst-case ISI and answer whether the channel's eye is open or not.

(c) Determine the sequence(s) for which the worst-case ISI occurs. Since all symbols occur with the same probability and since consecutive symbols are uncorrelated, all sequences are equally probable. An error occurs whenever the channel's eye is closed; calculate the error probability.

(d) Show that for an arbitrary choice of N and Δ the optimum solution $\mathbf{c}_{\text{opt}}$ in the sense of a *minimum mean-squared error* (MMSE) is the $(\Delta + 1)$-th column of the pseudo-inverse of the convolution matrix of $\mathbf{h}$, i.e., of

$$\left(\mathbf{H}^T\mathbf{H}\right)^{-1}\mathbf{H}^T$$

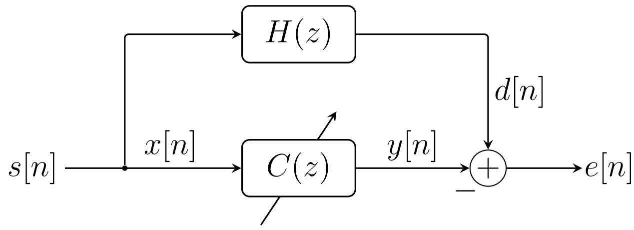
Make sure that you write down the matrix/vector dimensions explicitly!

(e) Now choose $N = 2$ and $\Delta = 0$ and determine the optimum filter coefficient vector $\mathbf{c}_{\text{opt}}$. Calculate the noise gain the equalizer would have. Can this equalizer open the channel's eye? If not, write down again the input sequence(s) for which an error occurs and compute the error probability.

(f) Repeat task **(e)** for $\Delta = 1$ and compare the results to those of **(e)**. Which delay Δ is preferable? Justify your answer!

Problem 3 (33 Points)

Consider the following system identification problem:



The input signal $s[n] = \sqrt{\frac{1}{2}}v[n] + \sqrt{\frac{1}{2}}v[n-1]$ which is composed of a with noise process $v[n]$ with zero mean and variance $\sigma_v^2 = 1$ is used as the input signal to identify the system $H(z) = \frac{1}{4} + 1z^{-1} + \frac{1}{2}z^{-2}$.

(a) Use the orthogonality criterion $\mathbb{E}\{e[n]\mathbf{x}[n]\} = \mathbf{0}$ to derive the Wiener-Hopf solution for an arbitrary coefficient vector $\mathbf{c}$.

(b) Compute the autocorrelation sequence $r_{ss}[k]$ of the input signal $s[n]$.

(c) Compute[2] the MSE-optimal filter coefficients $\mathbf{c}_{\text{opt}}$ of the adaptive filter $C(z)$ for $N = 2$ and $N = 3$. What requirements are necessary (i) to get a solution at all, (ii) to get the true filter coefficients of the unknown system? Are these conditions met in the cases $N = 2$ and $N = 3$? **Hint:** You might not even need to invert the 3×3 matrix.

(d) Show why the filter with $N = 2$ does not succeed in correctly identifying the system coefficients. You do not need to insert any actual numerical values, just examine a general case. Explain the influence of the statistical properties of the input signal on the result! **Hint:** Split the vectors into subvectors, e.g. $\mathbf{h} = \begin{bmatrix} \mathbf{h}_a^{\mathrm{T}} & \mathbf{h}_b^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}}$ with the according dimensions.

(e) Find an alternative input signal $s[n]$ that solves the problem of the adaptive filter with $N = 2$ coefficients and compute the new optimal coefficients $\mathbf{c}_{opt}$ (for $N = 2$).

[2] The inverse of a symmetric 3×3 Toeplitz-matrix is given as

$$\begin{bmatrix} a & b & c \\ b & a & b \\ c & b & a \end{bmatrix}^{-1} = \frac{1}{a^3 - 2ab^2 + 2b^2c - ac^2} \begin{bmatrix} a^2 - b^2 & bc - ba & b^2 - ac \\ bc - ba & a^2 - c^2 & bc - ba \\ b^2 - ac & bc - ba & a^2 - b^2 \end{bmatrix}.$$