Signal Processing and Speech Communication Laboratory—Prof. G. Kubin
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Exam *Adaptive Systems* on 2017/6/30

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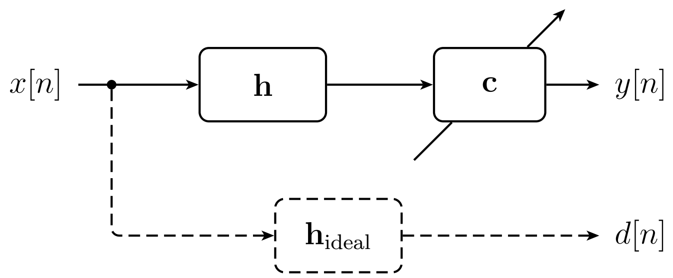
Exam duration: 3 hours
Maximum points: 100
Allowed material: mathematical formulary, calculator
You have to return this document after the exam.
Best wishes for a successful exam!

Problem 1 (34 Points)



We are given the impulse response vector $\mathbf{h} = [1,\ 0.5]^T$ of an LTI system. We further denote with $\mathbf{H}$ the *convolution matrix*, whose columns are composed of shifted vectors $\mathbf{h}$, i.e.,

$$\mathbf{H} = \begin{bmatrix} 1 & 0 & 0 & \cdots \\ 0.5 & 1 & 0 & \cdots \\ 0 & 0.5 & 1 & \cdots \\ 0 & 0 & 0.5 & \ddots \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix}.$$

We now want to design an FIR filter with coefficient vector $\mathbf{c}$ such that the cascade of $\mathbf{c}$ and $\mathbf{h}$ approximates the desired impulse respons vector $\mathbf{h}_{\text{ideal}}$. Let the residual vector as $\mathbf{r} = \mathbf{H}\mathbf{c} - \mathbf{h}_{\text{ideal}}$ then the Least-Squares (LS) solution is given by:

$$\mathbf{c}_{\text{LS}} = \arg\min_{\mathbf{c}} \|\mathbf{r}\|_2^2$$

where $\|\mathbf{r}\|_p$ is the p-norm operator applied on the vector $\mathbf{r}$.

(a) Show that the LS solution is given by

$$\mathbf{c}_{LS} = (\mathbf{H}^T\mathbf{H})^{-1}\mathbf{H}^T\mathbf{h}_{\text{ideal}}.$$

Use a general dimension $N = \dim(\mathbf{c})$ of the coefficient vector.

(b) Let $\mathbf{h}_{\text{ideal}}$ be such that $d[n] = x[n]$ (i.e., we design a delay-free equalizer). Compute the two-dimensional ($N = 2$) coefficient vector $\mathbf{c}$ optimal in the LS sense. Make sure that you use the correct dimensions for $\mathbf{h}_{\text{ideal}}$ and $\mathbf{H}$!

(c) Repeat task **(b)** for a three-dimensional ($N = 3$) coefficient vector[1].

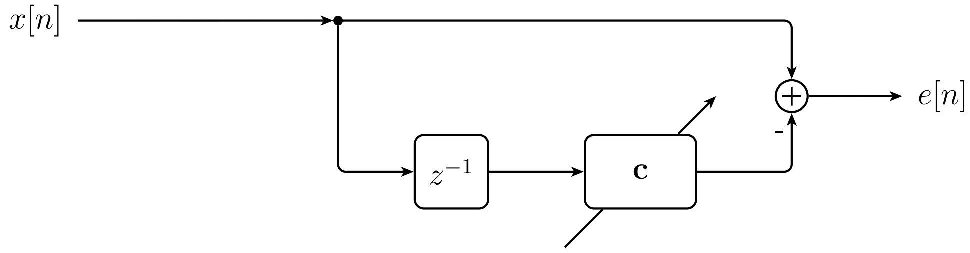
(d) Plot the pole-/zero diagram of the z-transform of the LTI system with coefficient vector $\mathbf{h}$. Can you invert this system? If yes, how could you accomplish this? What is the difference between the LS solution and the solution obtained by directly inverting the system?

[1]The inverse of a symmetric 3×3 Toeplitz-matrix is given as

$$\begin{bmatrix} a & b & c \\ b & a & b \\ c & b & a \end{bmatrix}^{-1} = \frac{1}{a^3 - 2ab^2 + 2b^2c - ac^2} \begin{bmatrix} a^2 - b^2 & bc - ba & b^2 - ac \\ bc - ba & a^2 - c^2 & bc - ba \\ b^2 - ac & bc - ba & a^2 - b^2 \end{bmatrix}.$$

Problem 2 (33 Points)

A prediction-error filter can be considered as a blind equalizer for AR processes.



Assume that the process $x[n]$ is obtained by passing a zero-mean, unit variance, white noise signal $v[n]$ through a filter specified by the following purely recursive difference equation:

$$x[n] = v[n] + 0.75x[n-1] - 0.25x[n-2].$$

(a) Calculate the autocorrelation sequence $r_{xx}[k]$ for $k = 0, 1, 2$.

(b) For a general N and a general autocorrelation sequence, use the Wiener-Hopf equation to find the MSE-optimal coefficient vector $\mathbf{c}$.

(c) For a predictor with $N = 1$ coefficient, find the MSE-optimal coefficient of the adaptive filter. Compute the minimum MSE.

(d) For a predictor with $N = 2$ coefficients, find the MSE-optimal coefficient vector $\mathbf{c}$ of the adaptive filter. Compute the minimum MSE. What can you observe?

Problem 3 (33 Points)

Transform domain adaptive filter: The performance surface of a transversal filter with input correlation matrix $\mathbf{R}$ may be written as

$$\zeta(v_0, v_1) = \zeta_{\min} + \mathbf{v}^T \mathbf{R} \mathbf{v}$$

where the vector $\mathbf{v} = [v_0 \quad v_1]^T$ is the difference between the filter tap-weight vector, $\mathbf{w} = [w_0 \quad w_1]^T$ and its optimum value, $\mathbf{w}_{\text{opt}}$, i.e., $\mathbf{v} = \mathbf{w} - \mathbf{w}_{\text{opt}}$. Consider a two-tap transversal filter that is characterized by the performance surface function

$$\zeta(v_0, v_1) = 0.1 + \mathbf{v}^T \mathbf{R} \mathbf{v}$$

where

$$\mathbf{R} = \begin{bmatrix} 1 & \alpha \\ \alpha & 1 \end{bmatrix}$$

Assuming that the filter input is a real-valued random process, answer the following questions:

(a) Find the points $(a, 0)$ and $(0, b)$ where the contour ellipse given by $\zeta(v_0, v_1) = 1.1$ meets the v_0 and v_1 axes and show that $a = b$. What can be implied about signal energy distributed among the taps of the filter?

(b) Define an orthogonal transformation matrix

$$\mathcal{T} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Show that the transformation $\mathbf{v}_{\mathcal{T}} = \mathcal{T}\mathbf{v}$, where $\mathbf{v} = [v_0 \quad v_1]^T$, is equivalent to rotating the coordinate axes v_0 and v_1 by θ radian counter-clockwise.

(c) Find the rotation angle θ that maximizes the ratio of the diagonal elements of $\mathbf{R}_{\mathcal{T}} = \mathcal{T}\mathbf{R}\mathcal{T}^T$. Is this result dependent on α?

(d) Noting that the diagonal elements of $\mathbf{R}_{\mathcal{T}}$ are the input signal energies after transformation, comment on your results in part (c).

(e) Show that for two-tap transversal filter with real-valued input processes, the optimum transformation matrix, $\mathcal{T}_{\text{opt}}$, is fixed and independent of the statistics of the underlying input process. What is $\mathcal{T}_{\text{opt}}$?