# Adaptive Systems UE (442.012)
# WS 18/19

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# Course Organization

- Problem class to accompany *Adaptive Systems VO* (442.011)
- **Problem class handout:**
  - https://www.spsc.tugraz.at/courses/adaptive-systems.html
- **3 homeworks:** 30-35 points each
  - 3 problems
  - 1-2 bonus problem(s)
  - Newsgroup: https://news.tugraz.at

# Before we start...

... a short reminder

Presentation of Bachelor Theses, Master Theses and Master Projects at the SPSC

- Friday, 18.10.2019
- at 15:00
- in the SPSC Seminarroom (IDEG134), ground floor, Inffeldgasse 16/c
- afterwards there will be pizza, beer and demos!

# Recap and roadmap for today

**Last Week:**

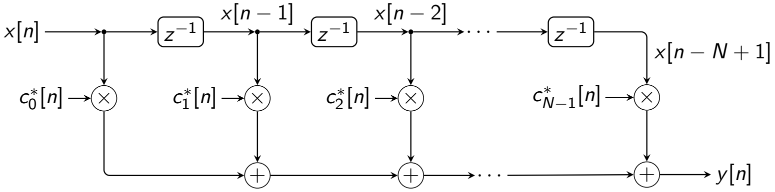
- *1.1* – LS optimal filtering
- *1.2* – Gradient Calculus

**This Week:**

- finish *1.2* – Gradient Calculus
- *1.3* – autocorrelation (expectation)
- *1.4* – Wiener Filtering (first try)

# Optimum Linear Filtering

**Adaptive FIR Filter:** $N$ coefficients and order $N-1$




$$y[n] = \sum_{k=0}^{N-1} c_k^*[n]x[n-k] = \underline{\mathbf{c}}^{\mathrm{H}}[n]\underline{\mathbf{x}}[n]$$

$c_k[n] = c_k \in \mathbb{R}\ \forall\ k$

$$= \sum_{k=0}^{N-1} c_k x[n-k] = \underline{\mathbf{c}}^{\mathrm{T}}\underline{\mathbf{x}}[n]$$

# Optimum Linear Filtering

**Adaptive FIR Filter:** $N$ coefficients and order $N-1$

$a$ ... scalar
$\underline{\mathbf{a}}$ ... vector
$\underline{\mathbf{A}}$ ... matrix

$$y[n] = \underline{\mathbf{c}}^{\mathsf{T}} \underline{\mathbf{x}}[n]$$

$$\underline{\mathbf{c}} = \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_{N-1} \end{bmatrix} = \begin{bmatrix} c_0 & c_1 & \cdots & c_{N-1} \end{bmatrix}^{\mathsf{T}}$$

$$\underline{\mathbf{x}}[n] = \begin{bmatrix} x[n] \\ x[n-1] \\ \vdots \\ x[n-N+1] \end{bmatrix} = \begin{bmatrix} x[n] & x[n-1] & \cdots & x[n-N+1] \end{bmatrix}^{\mathsf{T}}$$

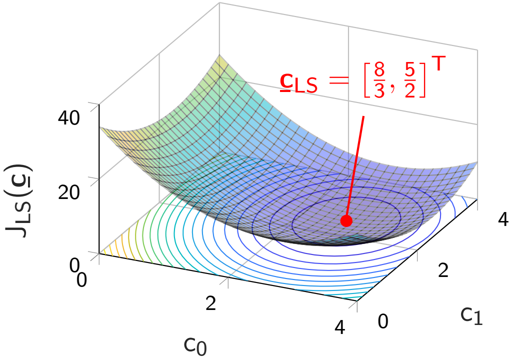
# Problem 1.1.

**LS-optimum filter:**

$$J_{\mathrm{LS}}(\underline{\mathbf{c}}) = \sum_{k=0}^{K-1} |e[k]|^2 = \underline{\mathbf{e}}^{\mathsf{T}} \underline{\mathbf{e}}$$

$$J_{\mathrm{min}} = J_{\mathrm{LS}}(\underline{\mathbf{c}}_{\mathrm{LS}}) = 0.1\dot{6}$$

$$\underline{\mathbf{e}} = \begin{bmatrix} e[0] \\ e[1] \\ \vdots \\ e[K-1] \end{bmatrix} \in \mathbb{R}^K$$



# Problem 1.2.

The previous problem has demonstrated that gradient calculus is important. To practice this calculus, determine $\underline{\boldsymbol{\nabla}}_{\underline{\mathbf{c}}} J(\underline{\mathbf{c}})$ for the following cost functions:

(i) $J(\underline{\mathbf{c}}) = K$

(ii) $J(\underline{\mathbf{c}}) = \underline{\mathbf{c}}^{\mathsf{T}}\underline{\mathbf{v}} = \underline{\mathbf{v}}^{\mathsf{T}}\underline{\mathbf{c}} = \langle \underline{\mathbf{c}}, \underline{\mathbf{v}} \rangle$

(iii) $J(\underline{\mathbf{c}}) = \underline{\mathbf{c}}^{\mathsf{T}}\underline{\mathbf{c}} = ||\underline{\mathbf{c}}||^2 = \langle \underline{\mathbf{c}}, \underline{\mathbf{c}} \rangle$

(iv) $J(\underline{\mathbf{c}}) = \underline{\mathbf{c}}^{\mathsf{T}}\underline{\mathbf{A}}\underline{\mathbf{c}}$, where $\underline{\mathbf{A}}^{\mathsf{T}} = \underline{\mathbf{A}}$ .

# Problem 1.2.

The previous problem has demonstrated that gradient calculus is important. To practice this calculus, determine $\underline{\boldsymbol{\nabla}}_{\underline{\mathbf{c}}} J(\underline{\mathbf{c}})$ for the following cost functions:

(i) $J(\underline{\mathbf{c}}) = K \quad \longrightarrow \quad \underline{\nabla}_{\underline{\mathbf{c}}} K = \underline{\mathbf{0}}$

(ii) $J(\underline{\mathbf{c}}) = \underline{\mathbf{c}}^{\mathsf{T}} \underline{\mathbf{v}} = \underline{\mathbf{v}}^{\mathsf{T}} \underline{\mathbf{c}} = \langle \underline{\mathbf{c}}, \underline{\mathbf{v}} \rangle \quad \longrightarrow \quad \underline{\nabla}_{\underline{\mathbf{c}}} \underline{\mathbf{c}}^{\mathsf{T}} \underline{\mathbf{v}} = \underline{\mathbf{v}}$

(iii) $J(\underline{\mathbf{c}}) = \underline{\mathbf{c}}^{\mathsf{T}} \underline{\mathbf{c}} = ||\underline{\mathbf{c}}||^2 = \langle \underline{\mathbf{c}}, \underline{\mathbf{c}} \rangle$

(iv) $J(\underline{\mathbf{c}}) = \underline{\mathbf{c}}^{\mathsf{T}} \underline{\mathbf{A}} \underline{\mathbf{c}}$, where $\underline{\mathbf{A}}^{\mathsf{T}} = \underline{\mathbf{A}}$ .

# MSE Optimal Filtering

## Wiener Filtering

When we assume that $x[n]$ and $d[n]$ are stochastic processes we can write a stochastic cost function as

$$J_{MSE}(\underline{\mathbf{c}}) = \mathrm{E}\left[|e[n]|^2\right] \text{ ... Mean Squared Error (MSE)}$$

and the optimum solution in the MSE sense is obtained as:

$$\underline{\mathbf{c}}_{MSE} = \operatorname{argmin}_{\underline{\mathbf{c}}} J_{MSE}(\underline{\mathbf{c}}).$$

# Problem 1.3.

The autocorrelation sequence of a stochastic process $x[n]$ is defined as

$$r_{xx}[n,k] := \mathrm{E}[x[n+k]x^*[n]]\,.$$

If $x[n]$ is *stationary*, then the autocorrelation sequence does not depend on time $n$, i.e.,

$$r_{xx}[k] = \mathrm{E}[x[n+k]x^*[n]]\,.$$

Calculate the autocorrelation sequence for the following signals with $A$ and $\theta$ are constant and $\varphi$ uniformly distributed as $\varphi \sim \mathcal{U}(-\pi,\pi]$. Are they stationary?

(i) $x[n] = A\sin(\theta n)$

(ii) $x[n] = A\sin(\theta n + \varphi)$

(iii) $x[n] = Ae^{j(\theta n+\varphi)}$

# MSE Optimal Filtering

## Wiener Filtering

We now assume $x[n]$ and $d[n]$ to be (jointly) stationary stochastic processes. The cost function is now stochastic

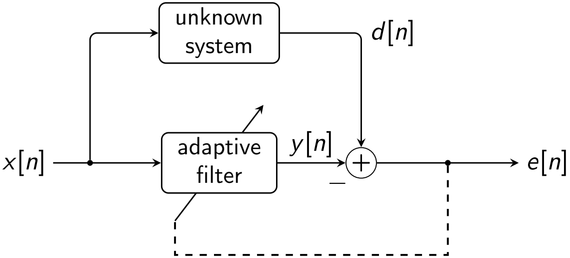
$$J_{\text{MSE}}(\underline{\mathbf{c}}) = \mathrm{E}\left[|e[n]|^2\right] \ldots \text{ Mean Squared Error (MSE)}$$

and the optimum solution in the MSE sense is obtained is found as

$$\underline{\mathbf{c}}_{\text{MSE}} = \text{argmin}_{\underline{\mathbf{c}}} J_{\text{MSE}}(\underline{\mathbf{c}}).$$

# MSE Optimal Filtering

**Problem 1.4.** For the optimum linear filtering problem, find $\underline{\mathbf{c}}_{\mathrm{MSE}}$ (i.e., the *Wiener-Hopf equation*). What statistical measurements must be known to get the solution?



# MSE Optimal Filtering

**Problem 1.5.** Assume that $x[n]$ and $d[n]$ are a jointly wide-sense stationary, zero-mean processes.

(i) Specify the autocorrelation matrix $\underline{\mathbf{R}}_{xx} = \mathrm{E}\left[\underline{\mathbf{x}}[n]\underline{\mathbf{x}}^{\mathsf{T}}[n]\right]$.

(ii) Specify the cross-correlation vector $\underline{\mathbf{p}} = \mathrm{E}[d[n]\underline{\mathbf{x}}[n]]$.

(iii) Assume that $d[n]$ is the output of a linear FIR filter to the input $x[n]$, i.e., $d[n] = \underline{\mathbf{h}}^{\mathsf{T}}\underline{\mathbf{x}}[n]$. Furthermore, $\dim(\underline{\mathbf{h}}) = \dim(\underline{\mathbf{c}})$. What is the optimal solution in the MSE sense?

# References


- **G. Moschytz and M. Hofbauer** (unfortunately in German only)
  "Adaptive Filter," *Springer Verlag*, Berlin Heidelberg, 2000
- **Simon Haykin**
  "Adaptive Filter Theory," Fourth Edition, *Prentice-Hall, Inc.*, Upper Saddle River, NJ, 2002
- **M. Vetterli, J. Kovacevic, and V. K. Goyal**
  "Foundations of Signal Processing," Cambridge University Press, 2014.