Signal Processing and Speech Communication Laboratory
Graz University of Technology

Exam for

442.001 / SES.208UF Signal Processing
442.009 Fundamentals of Discrete-Time Signals and Systems

taking place on 07.11.2025

Duration: 180 minutes, Achievable points: 100

Permitted aids: provided SPSC formulary, scientific (*non*-alphanumeric) calculator, supplied (*no* personally brought) paper, drinks and snacks.

! Important !

The problem statement and the formulary must be returned at the end of the exam!

Write your name and matriculation number on all sheets of your solution!

All results must be explained in a comprehensible way!

Problem 1 (25 Points)

A signal $x[n]$ where $0 \leq n < N$ should be represented as a combination of K base signals $\{b_k[n]\}_{k=0}^{K-1}$ where $0 \leq n < M$ with corresponding coefficients $\{a_k\}_{k=0}^{K-1}$ as

$$x[n] = \sum_{k=0}^{K-1} a_k \cdot b_k[n].$$

(a) In general, how would you choose the length M of the base signals? Discuss the cases (i) $M < N$, (ii) $M = N$ and (iii) $M > N$.

(b) How many base signals K are certainly sufficient to represent an arbitrary signal $x \in \mathbb{R}^N$ with $b_k \in \mathbb{R}^M$, for $k \in \{0 \dots K-1\}$ and real-valued coefficients $a_k \in \mathbb{R}$ with suitably chosen M? Using a simple example for $N = 2$, illustrate the case that a signal space cannot be sufficiently spanned with too few base signals.

(c) How can you compute the coefficients a_k for a general real valued signal x and orthonormal base signals b_k?

(d) Let $x[n]$ be given as

$$x[n] = \begin{cases} A & \text{if} \quad N_0 \leq n < N_1 \\ 0 & \text{otherwise} \end{cases}$$

where $0 < N_0 < N_1 \leq N$ and $b_k[n] = \delta[n-k]$. Determine the coefficients a_k.

(e) Which criteria must the basis vectors of an orthonormal basis obey? Is the basis in Task (d) an orthonormal basis? Prove!

Problem 2 (25 Points)

A discrete-time signal is obtained by sampling a continuous-time signal, yielding

$$x[n] = cos(2\pi \frac{f_1}{f_s} n + \frac{\pi}{4}) = x(\frac{n}{f_s}), \qquad n = 0, \ldots, N-1, \tag{1}$$

with sampling frequency f_s and analyzed with the N-point Discrete Fourier Transform (DFT) with $N = 4$.

(a) Compute the DFT frequencies $\theta[k]$ of the N-point DFT and list the discrete frequency bin center frequencies $f[k]$ depending on f_s for $k = [0, 1, 2, 3]$.

(b) Let $\frac{f_1}{f_s} = \frac{3}{8}$. Compute the 4-point DFT of $x[n]$ and sketch the magnitude spectrum $|X[k]|$ over $\theta[k]$. Why does spectral leakage occur here? Justify your answer!

(c) Let $\frac{f_1}{f_s} = \frac{5}{4}$. Again compute the 4-point DFT of $x[n]$ and sketch the magnitude spectrum $|X[k]|$ over $\theta[k]$. Why does aliasing occur here? Justify your answer!

(d) Derive an expression for all frequency ratios $\frac{f}{f_s}$ for $f_s = 4$ that produce the same DFT frequency θ as the ratio $\frac{f_1}{f_s} = \frac{5}{4}$.

Problem 3 (25 Points)

Consider the FIR filter with impulse response

$$h[n] = a\,\delta[n] + b\,\delta[n-1] + b\,\delta[n-2] + a\,\delta[n-3], \qquad a, b \in \mathbb{R}.$$

(a) Determine the transfer function $H(z)$ of the filter.

(b) Show that the system has a zero at $z = -1$ for arbitrary a, b.

(c) Determine all poles and zeros of $H(z)$ for arbitrary a, b.

(d) Design the coefficients a and b such that

- the frequency response $H(e^{j\theta}) = 1$ for $\theta_0 = 0$ and
- the frequency response $H(e^{j\theta})$ has a zero at $\theta_1 = \frac{2\pi}{3}$.

Provide the resulting values of a, b.

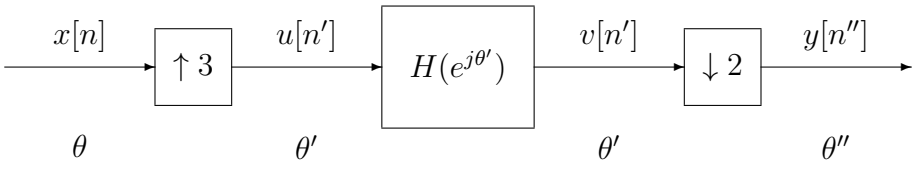
(e) For your design in the previous task (d), sketch the pole-zero diagram and identify the filter type (e.g. low-pass, high-pass, band-stop, band-pass). Justify your answer!

Problem 4 (25 Points)

Assume a discrete-time signal $x[n]$ has the discrete-time Fourier transform (DTFT)

$$X(e^{j\theta}) = \pi\delta_{2\pi}(\theta) \; + \; \pi e^{-j\frac{\pi}{4}}\delta_{2\pi}\Big(\theta - \frac{\pi}{2}\Big) \; + \; \pi e^{+j\frac{\pi}{4}}\delta_{2\pi}\Big(\theta + \frac{\pi}{2}\Big).$$

We process $x[n]$ through the multirate chain



where $H(e^{j\theta'})$ is an ideal low-pass filter with cutoff θ'_c (to be determined in part (d)).

(a) Use the inverse DTFT to find a closed-form expression for $x[n]$, and sketch $x[n]$ for $n = 0, 1, ..., 5$.

(b) Plot the magnitude $|X(e^{j\theta})|$ over at least two 2π-periods and mind the correct amplitude scaling. State whether the spectrum is even/odd and what implications this has on $x[n]$.

(c) Give a time-domain description of $u[n']$. Then derive and sketch $|U(e^{j\theta'})|$ over two periods and mind the correct amplitude scaling. Clearly label the locations of the spectral lines and their spacing.

(d) Assume $v[n'] = u[n']$ (i.e., temporarily remove the filter $H(e^{j\theta'})$). Sketch the magnitude spectrum $|Y(e^{j\theta''})|$ after downsampling by 2 and mind the correct amplitude scaling. Clearly indicate any aliasing effects if they occur.

(e) Now determine the largest cutoff frequency θ'_c of the ideal low-pass filter $H(e^{j\theta'})$ such that, after downsampling, no aliasing occurs. Provide an inequality describing the allowed range of θ'_c in terms of the up - and downsampling factors.

(f) With this filter in place, sketch $|V(e^{j\theta'})|$ (after $H(e^{j\theta'})$) and then sketch the resulting $|Y(e^{j\theta''})|$ after downsampling by 2 and mind the correct amplitude scaling. Assume that the passband gain of $H(e^{j\theta'})$ equals 1.