Signal Processing and Speech Communication Laboratory
Graz University of Technology

Exam for

442.001 / SES.208UF Signal Processing
442.009 Fundamentals of Discrete-Time Signals and Systems

taking place on 10.07.2026

Duration: 180 minutes, Achievable points: 100

Permitted aids: provided SPSC formulary, scientific (*non*-alphanumeric) calculator, supplied (*no* personally brought) paper, drinks and snacks.

! Important !

The problem statement and the formulary must be returned at the end of the exam!

Write your name and matriculation number on all sheets of your solution!

All results must be explained in a comprehensible way!

Problem 1 (25 Points)

The base $\mathcal{B}$ is spanned by K base signals $b_k[n] \in \mathbb{R}$ of the form

$$b_k[n] = \Re e\left\{e^{j\frac{\pi}{4}kn}\right\}$$

(with $k = 0, 1, \ldots, K-1$ and $n = -\frac{N}{2}, -\frac{N}{2}+1, \ldots, \frac{N}{2}-2, \frac{N}{2}-1$). Let $K = 3$ and $N = 8$.

(a) Draw the basis signals and show that it is an orthogonal basis.

(b) Calculate the corresponding ortho*normal* basis $\tilde{b}_k[n]$.

(c) A signal $x[n]$ is to be represented as the sum of K **orthonormal** basis signals $\{\tilde{b}_k[n]\}_{k=0}^{K-1}$

$$x[n] = \sum_{k=0}^{K-1} a_k \cdot \tilde{b}_k[n].$$

Derive how to determine the coefficients a_k for an arbitrary signal $x[n]$.

(d) Let $x[n]$ be given as

$$x[n] = \begin{cases} A & \text{if} \quad n = -\frac{N}{2}, -\frac{N}{2}+1, \ldots, \frac{N}{2}-2, \frac{N}{2}-1 \\ 0 & \text{otherwise} \end{cases}$$

Determine the coefficients a_k. Provide all necessary computation steps.

(e) Can the given base $\mathcal{B}$ span the signal $x[n] = \cos\left(\frac{3\pi}{2}n\right)$? If not, why and what would you have to change to make it possible?

Problem 2 (25 Points)

A discrete-time signal is obtained by sampling a continuous-time signal, yielding

$$x[n] = \cos\left(2\pi \frac{f_1}{f_s} n + \frac{\pi}{4}\right) = x\left(\frac{n}{f_s}\right), \qquad n = 0, \ldots, N-1, \tag{1}$$

with sampling frequency f_s and analyzed with the N-point Discrete Fourier Transform (DFT) with $N = 4$.

(a) Compute the DFT frequencies $\theta[k]$ of the N-point DFT for $k \in \{0, 1, 2, 3\}$.

(b) Let $\frac{f_1}{f_s} = \frac{3}{8}$. Compute the 4-point DFT of $x[n]$ and sketch the magnitude spectrum $|X[k]|$ over $\theta[k]$. Why does spectral leakage occur here? Justify your answer!

(c) Let $\frac{f_1}{f_s} = \frac{5}{4}$. Again compute the 4-point DFT of $x[n]$ and sketch the magnitude spectrum $|X[k]|$ over $\theta[k]$. Why does aliasing occur here? Justify your answer!

(d) Derive an expression for all frequency ratios $\frac{f}{f_s}$ for $f_s = 4$ that produce the same DFT frequency θ as the ratio $\frac{f_1}{f_s} = \frac{5}{4}$.

Problem 3 (25 Points)

Consider two two-sided sequences $h[n]$ and $g[n]$. A new signal $x[n]$ is built by interleaving them sample-by-sample:

$$x[n] = \ldots, h[-2], g[-2], h[-1], g[-1], h[0], g[0], h[1], g[1], h[2], g[2], \ldots,$$

with $x[0] = h[0]$.

(a) Express $X(z)$ in terms of $H(z)$ and $G(z)$.

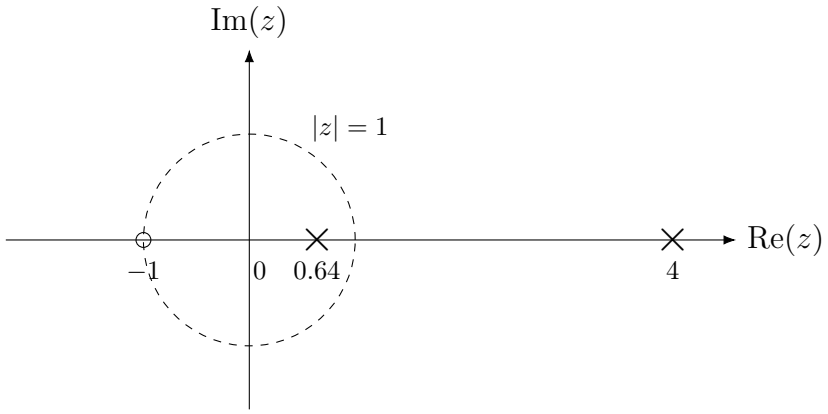
Hint: Split the sum $X(z) = \sum_n x[n]z^{-n}$ into its even and odd parts. Can they be expressed in terms of $h[n]$ and $g[n]$?

(b) $H(z)$ and $G(z)$ each have two poles on the positive real axis, with the following regions of convergence (ROCs):

$$\text{ROC}_H : \ 0.64 < |z| < 4, \qquad \text{ROC}_G : \ 0.25 < |z| < 9.$$

Compute the ROC of $X(z)$ and sketch the poles of $X(z)$ in the z-plane.

(c) Below is the pole-zero plot of $H(z)$.



(i) For each of the three possible interpretations of this pole-zero plot (causal, anti-causal, and acausal), state whether the resulting system would be stable. Explain your reasoning.

(ii) What kind of filter does this system represent (lowpass, highpass, bandpass, or other)? Justify your answer based on the pole-zero plot.

(d) Using the pole-zero plot from task (c):

(i) Write $H(z)$ as a ratio of polynomials in z.

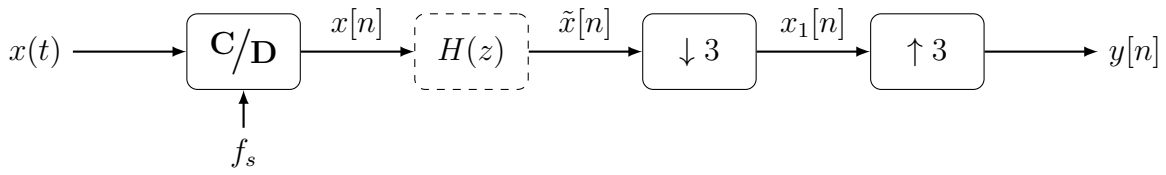
(ii) Using this $H(z)$, compute $|H(e^{j\theta})|$ for $\theta \in \{0, \frac{\pi}{2}, \pi\}$.

(iii) Suppose we want to redesign the system so that $|H(e^{j\theta})|_{\theta=\frac{\pi}{2}} = 0$, while keeping $h[n]$ real-valued. Derive the new $H(z)$ and verify the magnitude at $\theta = \frac{\pi}{2}$.

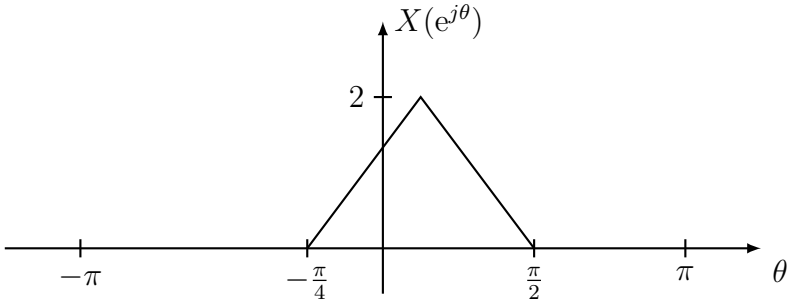
Hint: Is a single zero enough to put a null at $\theta = \frac{\pi}{2}$, while staying real-valued?

Problem 4 (25 Points)

A continuous-time signal $x(t)$ is sampled and processed by the multirate system shown below. The block $H(z)$ (dashed) is an *optional* low-pass filter.



The spectrum $X(\mathrm{e}^{j\theta})$ of the sampled signal $x[n]$ is given in the figure below.



Unless stated otherwise, use a sampling rate of $f_s = 16$ kHz.

All spectra in this problem are discrete-time Fourier transforms (DTFT). Whenever a task asks you to sketch a spectrum, draw it over two full periods, $\theta \in [-2\pi, 2\pi]$, mark the relevant frequencies on the θ-axis, and pay attention to the amplitude scaling.

(a) Is the time signal $x[n]$ real-valued, imaginary, symmetric or complex-valued? Justify your answer using the spectrum. For the given sampling rate $f_s = 16$ kHz, what is the highest (analog) frequency contained in the signal? Compute the bandwidth B and the center frequency θ_c of the spectrum.

(b) With the filter absent (i.e. $\tilde{x}[n] = x[n]$), sketch $X_1(\mathrm{e}^{j\theta'})$ and $Y(\mathrm{e}^{j\theta})$. Does aliasing occur? Explain!

(c) Derive the general condition relating the signal bandwidth B and the downsampling factor M such that no aliasing occurs.

(d) We noticed that there is no useful information above $f_{c,\mathrm{LP}} = 8/3$ kHz. Compute the corresponding digital cutoff frequency $\theta_{c,\mathrm{LP}}$ and design an ideal low-pass filter $H(z)$, i.e. $H(\mathrm{e}^{j\theta}) = 1$ for $|\theta| \leq \theta_{c,\mathrm{LP}}$ and $H(\mathrm{e}^{j\theta}) = 0$ otherwise, that removes everything above this threshold.

(e) Sketch $H(\mathrm{e}^{j\theta})$ and redraw the complete chain with $H(z)$ inserted, hence $\tilde{X}(\mathrm{e}^{j\theta})$, $X_1(\mathrm{e}^{j\theta'})$ and $Y(\mathrm{e}^{j\theta})$. Does aliasing occur? Compare the output $Y(\mathrm{e}^{j\theta})$ to the original $X(\mathrm{e}^{j\theta})$ and explain the effect of the low-pass filter.