Signal Processing and Speech Communication Laboratory
Graz University of Technology

Exam for

442.001 / SES.208UF Signal Processing
442.009 Fundamentals of Discrete-Time Signals and Systems

taking place on 30.09.2026

Duration: 180 minutes, Achievable points: 100

Permitted aids: provided SPSC formulary, scientific (*non*-alphanumeric) calculator, supplied (*no* personally brought) paper, drinks and snacks.

! Important !

The problem statement and the formulary must be returned at the end of the exam!

Write your name and matriculation number on all sheets of your solution!

All results must be explained in a comprehensible way!
Final answers without the calculation steps (even if correct) receive no points.

Problem 1 (25 Points)

A signal $x[n]$ where $0 \leq n < N$ should be represented as a combination of K base signals $\{b_k[n]\}_{k=0}^{K-1}$ where $0 \leq n < M$ with corresponding coefficients $\{a_k\}_{k=0}^{K-1}$ as

$$x[n] = \sum_{k=0}^{K-1} a_k \cdot b_k[n].$$

(a) In general, how would you choose the length M of the base signals? Discuss the cases (i) $M < N$, (ii) $M = N$ and (iii) $M > N$.

(b) How many base signals K are certainly sufficient to represent an arbitrary signal $x \in \mathbb{R}^N$ with $b_k \in \mathbb{R}^M$, for $k \in \{0 \dots K-1\}$ and real-valued coefficients $a_k \in \mathbb{R}$ with suitably chosen M? Using a simple example for $N = 2$, illustrate the case that a signal space cannot be sufficiently spanned with too few base signals.

(c) From now on, we assume that $N = M$. How can you compute the coefficients a_k for a general real-valued signal x and orthonormal base signals b_k?

(d) Let $x[n]$ be given as

$$x[n] = \begin{cases} A & \text{if} \quad N_0 \leq n < N_1 \\ 0 & \text{otherwise} \end{cases}$$

where $0 < N_0 < N_1 \leq N$ and $b_k[n] = \delta[n-k]$. Determine the coefficients a_k.

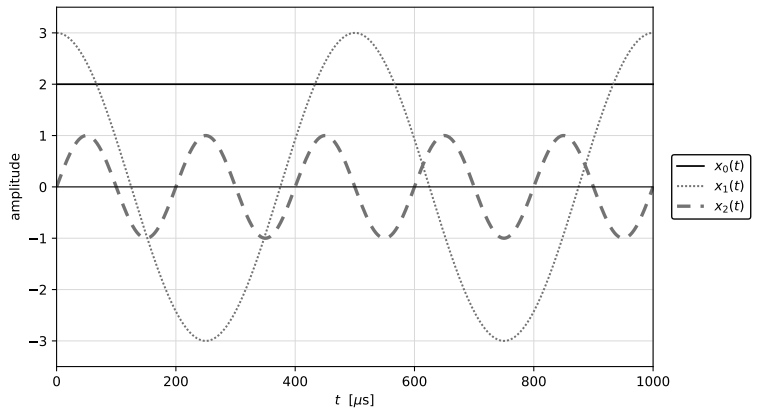
(e) Which criteria must the base signals of an orthonormal basis obey? Is the basis in Task (d) an orthonormal basis? Prove!

Problem 2 (25 Points)

Consider a continuous-time signal that is the sum of three components,

$$x(t) = x_0(t) + x_1(t) + x_2(t),$$

where $x_0(t)$ is a constant (DC) component and $x_1(t)$, $x_2(t)$ are sinusoids. The three components are plotted *individually* in the figure (not their sum). The signal is sampled at $f_s = 16$ kHz to obtain the discrete-time signal $x[n] = x(n/f_s)$.



(a) From the figure, determine the amplitude and frequency of each component. Then write down $x_0(t)$, $x_1(t)$, $x_2(t)$ as explicit closed-form expressions (each sinusoid as either a cosine or a sine, as appropriate).

(b) Determine the smallest N that is a power of two such that sampling $x_1(t)$ and $x_2(t)$ at $f_s = 16$ kHz over a window of N samples produces DFT coefficients with *no spectral leakage*. Concretely, this means f_1 and f_2 must each fall exactly on an integer DFT bin index:

$$L_i = \frac{f_i \cdot N}{f_s} \in \{1, 2, \ldots, \tfrac{N}{2} - 1\}, \qquad i = 1, 2.$$

Do not simply assume a value for N — derive it.

> **If you could not solve this part:** use $N = 16$, $L_1 = 2$, $L_2 = 5$ for the remaining sub-tasks.

(c) Using the value of N found in (b), write out the sampled signal

$$x[n] = x_0[n] + x_1[n] + x_2[n], \qquad n = 0, 1, \ldots, N-1$$

explicitly, i.e. substitute the amplitudes and the bin indices L_1, L_2 you found. State L_1, L_2, and the frequency resolution $\Delta f = f_s/N$.

(d) Compute the DFT coefficients $X[k]$, $k = 0, \ldots, N-1$, of the sampled signal $x[n]$ using the DFT pairs given on your formula sheet. Present your answer as a list (or table) of the nonzero values of $X[k]$.

Hint: Use the formula sheet.

(e) Solve the following sub-tasks:

(i) Verify explicitly, using one k for which X is nonzero as an example, that the Hermitian symmetry property $X[N-k] = X^*[k]$ holds, as expected for a real-valued signal $x[n]$.

(ii) Suppose $x_2(t)$ had instead been a *cosine* of the same amplitude and frequency (rather than a sine). Which entries of $X[k]$ would change, and how?

Problem 3 (25 Points)

We obtain the step response

$$y[n] = \left(\frac{1}{2}\right)^{n-1} u[n+1]$$

of an LTI system $\mathcal{H}$ for the input signal $x[n] = u[n]$, where $u[n]$ is the unit step function.

(a) Calculate the transfer function $H(z)$ and plot its pole-zero diagram.

(b) Find the impulse response $h[n]$ (e.g. via the inverse z-transform of $H(z)$, or directly from the relationship between step response and impulse response). Plot $h[n]$ for $-5 \leq n \leq 5$.

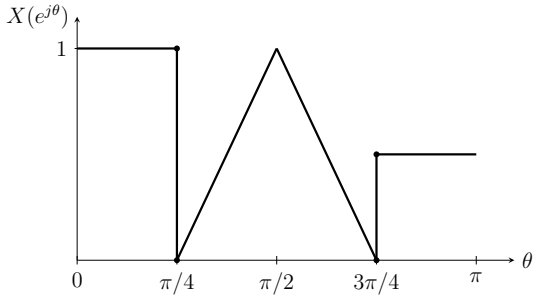
(c) Is the system stable? Is the system causal? Explain your reasoning.

(d) Sketch the magnitude frequency response of the system. Does the system implement a high-pass, low-pass or band-pass?

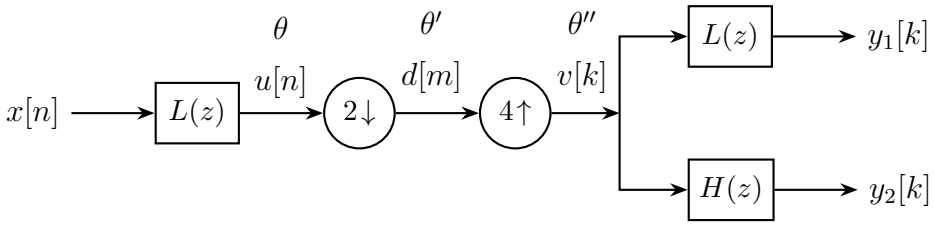
(e) Compute the system's response to the constant input signal $x[n] = c$ with $c \in \mathbb{R}$. Is the system invertible? Explain your reasoning.
Hint: For system to be invertible, it is necessary that 2 arbitrary distinct input signals always generate 2 distinct output signals.

Problem 4 (25 Points)

Consider a discrete-time signal $x[n]$ with the following symmetric frequency spectrum $X(e^{j\theta})$:



Now, consider the following multirate processing scheme in which $L(z)$ is an ideal lowpass filter with cutoff frequency $\theta_c = \pi/2$ and $H(z)$ is an ideal highpass filter with cutoff frequency $\theta_c = \pi/2$:



(a) Let $u[n]$ denote the signal at the output of the first filter $L(z)$. Sketch its spectrum $U(e^{j\theta})$ for $\theta \in (-\pi, \pi]$.

(b) Solve/answer the following sub-tasks:

(i) State the general condition (in terms of the decimation factor M) under which downsampling a signal that is bandlimited to $|\theta| \leq \theta_c$ by a factor of M introduces *no* aliasing.

(ii) Using this condition, argue whether downsampling $u[n]$ by 2 causes aliasing, and explain in what sense this is a *critical (borderline) case*.

(iii) Finally: if the cutoff frequency of $L(z)$ were increased slightly beyond $\pi/2$ (with $H(z)$ unchanged), would the downsampling $\downarrow 2$ operation now introduce aliasing? Justify your answer briefly.

(c) Let $d[m]$ denote the signal after downsampling $u[n]$ by 2. Sketch $D(e^{j\theta'})$ for $\theta' \in (-\pi, \pi]$.

(d) Let $v[k]$ denote the signal after upsampling $d[m]$ by 4. Sketch $V(e^{j\theta''})$ for $\theta'' \in (-\pi, \pi]$.

(e) Sketch the two output spectra $Y_1(e^{j\theta''})$ and $Y_2(e^{j\theta''})$ for $\theta'' \in (-\pi, \pi]$.