Signal Processing and Speech Communication Laboratory
Graz University of Technology

Exam for

442.001 / SES.208UF Signal Processing
442.009 Fundamentals of Discrete-Time Signals and Systems

taking place on 26.03.2026

Duration: 180 minutes, Achievable points: 100

Permitted aids: provided SPSC formulary, scientific (*non*-alphanumeric) calculator, supplied (*no* personally brought) paper, drinks and snacks.

! Important !

The problem statement and the formulary must be returned at the end of the exam!

Write your name and matriculation number on all sheets of your solution!

All results must be explained in a comprehensible way!

Problem 1 (25 Points)

Consider the following three discrete-time signals of length 4:

$$b_k[n] = \frac{1}{2} e^{j\frac{\pi}{2}kn}, \quad k \in \{0, 1, 2\}, \quad n = 0, 1, 2, 3.$$

Let

$$\mathcal{B} = \{b_0, b_1, b_2\}$$

and consider the two signals

$$x[n] = (\, 1 + j,\ -1,\ 1 - j,\ 1\,), \qquad y[n] = (\, 1,\ 1,\ 0,\ 1\,).$$

(a) Show that the signals in $\mathcal{B}$ are orthonormal, and conclude that $\mathcal{B}$ is an orthonormal basis for a 3-dimensional subspace of $\mathbb{C}^4$.

(b) Prove that $x[n]$ can be represented by the basis signals in $\mathcal{B}$ and compute the corresponding coefficients α_k. **Note**: Guessing the coefficients is not sufficient.

(c) Prove that $y[n]$ cannot be represented by the basis signals in $\mathcal{B}$.

(d) Extend the set $\mathcal{B}$ by one additional signal $b_3[n]$, such that

1. the new set $\mathcal{B}_{new} = \{b_0, b_1, b_2, b_3\}$ is an is an orthonormal basis of $\mathbb{C}^4$, and
2. $y[n]$ can be represented with respect to $\mathcal{B}_{new}$.

(e) Verify that $\mathcal{B}_{new} = \{b_0, b_1, b_2, b_3\}$ is *orthonormal*.

Remark: A signal $s[n]$ can be represented by a set of basis signals $b_k[n]$, iff there exist coefficients α_k, such that

$$s[n] \stackrel{!}{=} \sum_k \alpha_k b_k[n].$$

Problem 2 (25 Points)

Consider the double-sided discrete-time signal $x[n] = \cos(\theta_0 n) \cdot w[n]$, where the rectangular window of length N is defined as

$$w[n] = \begin{cases} 1 & \text{if} \quad 0 \leq n < N, \\ 0 & \text{otherwise.} \end{cases}$$

Thus, the observed signal is a finite-length segment of a periodic cosine. Let $\theta_0 = \frac{\pi}{2}$ and $N = 4$.

(a) Compute the DTFT $X(e^{j\theta})$ of the windowed cosine. You may use the formulary. Sketch the magnitude $|X(e^{j\theta})|$ for $\theta \in [-\pi, \pi]$.

(b) Compute the 4-point DFT $X[k]$ of $x[n]$ for $0 \leq n < 4$. You may use the formulary. Sketch the magnitude $|X[k]|$.

(c) Compute the 5-point DFT of the zero-padded signal

$$\tilde{x}[n] = \begin{cases} x[n] & \text{if} \quad n < 4, \\ 0 & \text{otherwise,} \end{cases}$$

for $0 \leq n < 5$ *without* using the formulary. Thus, the window length remains $N = 4$, but the DFT size is now 5. Sketch the magnitude $|\tilde{X}[k']|$. Hint:

$$\sum_{n=0}^{N-1} q^n = \frac{1 - q^N}{1 - q},$$

(d) Sketch now the DTFT $|X(e^{j\theta})|$ and the DFTs $|X[k]|$ and $|\tilde{X}[k']|$ for $\theta \in [-\pi, \pi]$ in the same plot, i.e., you need to map the DFT indices k and k' to their corresponding frequencies θ.

(e) Explicitly state the basis functions of the DTFT and DFT transforms you computed above. Explain conceptually how the DFT and DTFT are related in terms of their respective signal spaces and basis functions. In your explanation, you may refer to the unit circle.

Problem 3 (25 Points)

Consider a system with impulse response

$$h[n] = (2^n - (-2)^n) \cdot u[-n]$$

(a) Sketch the impulse response of this system for $-5 \leq n < 5$ for $n \in \mathbb{Z}$. Is the system causal, acausal, or anti-causal? Explain your answer.

(b) Compute the Z-transform $H(z)$ of the impulse response *without* the use of transform pairs/identities from the formula. Hint: $\sum_{n=0}^{\infty} q^n = \frac{1}{1-q}$ for $|q| < 1$.

(c) Determine the poles/zeros and the region of convergence of $H(z)$. Hint: your derivations from task (b) may be insightful. Sketch the pole-zero plot. Is the system stable? Explain your answer.

(d) Given your pole/zero plot, determine the frequencies θ where $|H(e^{j\theta})|$ has its maxima and minima. Explain your answer. In case you could not solve task (b) and task (c), assume that $H(z)$ has poles at $z_\infty \in \{-2, 2\}$.

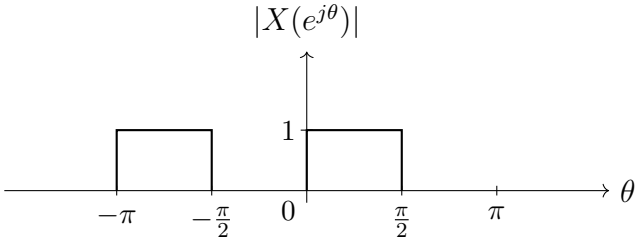
(e) Determine further the specific values $\max_\theta |H(e^{j\theta})|$ and $\min_\theta |H(e^{j\theta})|$, and sketch $|H(e^{j\theta})|$ for $\theta \in [-\pi, \pi]$.

Problem 4 (25 Points)

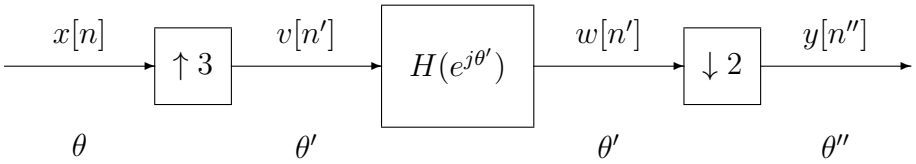
Assume that a discrete-time signal $x[n]$ has the discrete-time Fourier transform (DTFT) $X(e^{j\theta})$. The DTFT's magnitude spectrum $|X(e^{j\theta})|$ is given by

$$|X(e^{j\theta})| = \begin{cases} 1, & \theta \in \left[-\pi, -\frac{\pi}{2}\right] \cup \left[0, \frac{\pi}{2}\right] \\ 0, & \text{otherwise} \end{cases} \qquad \text{for } \theta \in [-\pi, \pi].$$

and shown below:



Consider the following multirate system:



The filter $H(e^{j\theta'})$ is an ideal low-pass filter with gain 3:

$$H(e^{j\theta'}) = \begin{cases} 3, & |\theta'| \leq \theta'_c \\ 0, & \text{otherwise.} \end{cases}$$

Note: When sketching the spectra, pay attention to the correct amplitude scaling and clearly label the frequency locations, interval boundaries, and periodic repetitions.

(a) Check wether $|X(e^{j\theta})|$ is an even function of θ. What conclusion can you draw about wether $x[n]$ is real- or complex-valued?

(b) Sketch the magnitude spectrum $|V(e^{j\theta'})|$ after upsampling the signal by 3 for $\theta' \in [-\pi, \pi]$.

(c) Assume temporarily that $H(e^{j\theta'}) = 1 \ \forall \ \theta'$, i.e. $w[n'] = v[n']$. Sketch $|Y(e^{j\theta''})|$ after downsampling by 2 for $\theta'' \in [-2\pi, 2\pi]$. Based on your sketch, state whether the original spectrum can be recovered uniquely after the downsampling step, or whether aliasing causes irreversible information loss. Briefly explain.

(d) Now determine the largest cutoff frequency θ'_c of the ideal low-pass filter $H(e^{j\theta'})$ such that

(1) the spectral images created by the upsampling by 3 are suppressed, and

(2) after downsampling by 2, no aliasing occurs. Express θ_c' in terms of the upsampling factor L and downsampling factor M.

(e) Now assume that the filter $H(e^{j\theta'})$ from task (d) is used. Sketch $|W(e^{j\theta'})|$ after filtering and then sketch the resulting $|Y(e^{j\theta''})|$ after downsampling by 2. Sketch both spectra for $\theta'' \in [-2\pi, 2\pi]$. In this case, does the downsampling step cause any information loss? Briefly explain.