Lesson 3: Fading Memory Nonlinearities

Nonlinear Signal Processing – SS 2019

Christian Knoll
Signal Processing and Speech Communication Laboratory
Graz University of Technology

May 23, 2019

Session contents

- Today:
  - Volterra Series as representation of fading-memory NL
  - System identification using different inputs
  - Time series modelling (Homework)
- Next time:
  - Higher-order statistics and spectral analysis

Volterra series – Definition

- A finite Volterra series of order p and memory length M:

$$
\begin{aligned}
y[n] = h_0 &+ \sum_{m_1=0}^{M-1} h_1[m_1]\, x[n-m_1] \\
&+ \sum_{m_1=0}^{M-1} \sum_{m_2=0}^{M-1} h_2[m_1, m_2]\, x[n-m_1]\, x[n-m_2] + \\
&+ \sum_{m_1=0}^{M-1} \sum_{m_2=0}^{M-1} \sum_{m_3=0}^{M-1} h_3[m_1, m_2, m_3]\, x[n-m_1]\, x[n-m_2]\, x[n-m_3] + \cdots + \\
&+ \sum_{m_1=0}^{M-1} \cdots \sum_{m_p=0}^{M-1} h_p[m_1, \ldots, m_p] \prod_{i=1}^{p} x[n-m_i]
\end{aligned}
$$

- *Universal approximator* for time-invariant causal operators with fading memory (for bounded input signals)

Volterra series – System identification (1)

- Only term with $h_1[\cdot]$ is linear!
- Complexity grows exponentially with M^p
- Choice of input signal for system identification?
- Remember RBF-fits: Model also linear in coefficients $\rightarrow$ Least-squares fit easy

$$y[n] = \sum_k \alpha_k \phi_k(x[n]) \quad \text{N equations}$$

- Arrange in equation system

$$\underbrace{\begin{bmatrix} y[0] \\ \vdots \\ y[N-1] \end{bmatrix}}_{\boldsymbol{y}} = \underbrace{\begin{bmatrix} \phi_1(x[0]) & \cdots & \phi_K(x[0]) \\ \vdots & \ddots & \vdots \\ \phi_1(x[N-1]) & \cdots & \phi_K(x[N-1]) \end{bmatrix}}_{\boldsymbol{\Phi},\ (N\times K),\ N\gg K} \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_K \end{bmatrix}$$

Volterra series – System identification (2)

- Here we use the same method, just different basis functions
- Second order Volterra system as example $(1 + M + M^2$ coeff.)
- You need "quite some" data!

$$y[n] = h_0 + \sum_{m_1=0}^{M-1} h_1[m_1]\, x[n-m_1] + \sum_{m_1=0}^{M-1} \sum_{m_2=0}^{M-1} h_2[m_1, m_2]\, x[n-m_1]\, x[n-m_2]$$

$$\begin{bmatrix} y[0] \\ y[1] \\ \vdots \\ y[N-1] \end{bmatrix} = \begin{bmatrix} 1 & x[0] & \cdots & x[-M+1] & x^2[0] & x[0]x[-1] & \cdots & x^2[-M+1, -M+1] \\ 1 & x[1] & \cdots & x[-M+2] & x^2[1] & x[1]x[0] & \cdots & x^2[-M+2, -M+2] \\ \vdots & & & & & & & \vdots \end{bmatrix} \begin{bmatrix} h_0 \\ h_1[0] \\ \vdots \\ h_1[M-1] \\ h_2[0,0] \\ h_2[0,1] \\ \vdots \\ h_2[M-1, M-1] \end{bmatrix}$$

- Basis vectors/functions are the data products

Volterra series – Matlab files

- Function `vkernels.m` does the LS-fit (use `p3_1.m` as tutorial)
- Nonlinearities defined in `nlsystem1.m`

$$y[n] = a_0 x[n] + a_1 x[n-1] + a_2 x[n]^2 + a_3 x[n] x[n-1]$$

- Output of `vkernels.m` for this nonlinearity:

$$\texttt{H\{1\}} = 0$$

$$\texttt{H\{2\}} = \begin{bmatrix} a_0 & a_1 \end{bmatrix}^T$$

$$\texttt{H\{3\}} = \begin{bmatrix} \overbrace{a_2}^{x[n]^2} & \overbrace{a_3}^{x[n]x[n-1]} \\ \underbrace{a_3}_{x[n-1]x[n]} & \underbrace{0}_{x[n-1]^2} \end{bmatrix}$$

- Or optionally a structure `Vmodel` that can be directly passed to `vkernels_o.m` for Problem 3.2

Volterra series – Time series modelling/forecasting

- Given a correlated (not necessarlily just second order!) time series $\boldsymbol{s} = [s_1, \ldots, s_N]^T$
- Current sample s_n depends on past samples
- Volterra series one way to model the dependence
- Input $\boldsymbol{x}$ and output $\boldsymbol{y}$ both generated from $\boldsymbol{s}$
- Partitioning of $\boldsymbol{s}$ into training and validation sequences

$\rightarrow$ Homework

Higher order statistics and spectral analysis

- We are used to first- and second-order statistics
  - Mean: $\mu = \mathrm{E}\{x[n]\}$
  - ACF: $m_2[k] = \mathrm{E}\{x[n]x[n+k]\}$
- This is suitable as long as we deal with linear systems, e.g.

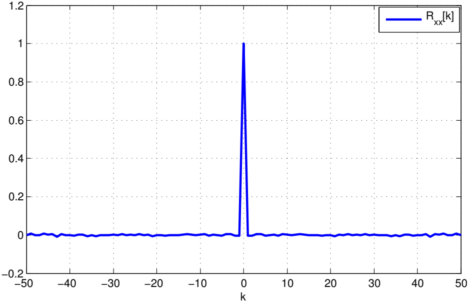
$$y[n] = \sum_{k=0}^{K} x[k]h[n-k]$$

- Standard example for a random process: *Linear process*, i.e. $x[k]$ is a white-noise process driving a linear system
- But what if our system is nonlinear, e.g.

$$y[n] = a_0x[n] + a_1x[n-1] + a_2x[n]^2 + a_3x[n]x[n-1]$$

Higher order statistics and spectral analysis - Example

- Linear system w. $K = 10$, driven by white noise

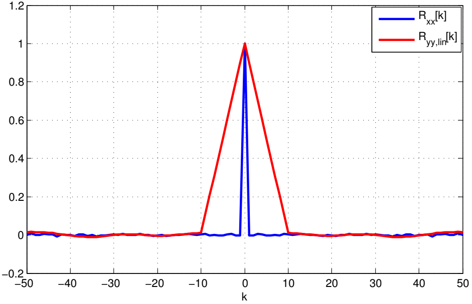


-
-

$\rightarrow$

Higher order statistics and spectral analysis - Example

- Linear system w. $K = 10$, driven by white noise

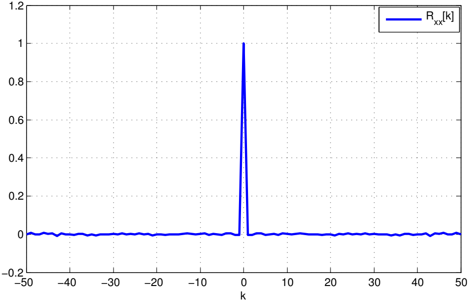


- Support of non-zero ACF indication for memory
-

$\rightarrow$

Higher order statistics and spectral analysis - Example

- Linear system w. $K = 10$, driven by white noise

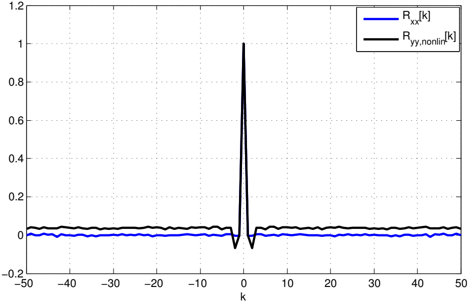


- Support of non-zero ACF indication for memory
- How do you evaluate nonlinear combinations of input samples?

$\rightarrow$

Higher order statistics and spectral analysis - Example

- Linear system w. $K = 10$, driven by white noise



- Support of non-zero ACF indication for memory
- How do you evaluate nonlinear combinations of input samples?

$\rightarrow$ Evaluate HOS, e.g. third-order $\mathrm{E}\{x[n]x[n+k]x[n+l]\}$

HOSA – Definitions

- Generalization of ACF: Non-central moments of order r for stationary process $x[n]$

$$m_r[k_1, \ldots, k_{r-1}] = \mathsf{E}\{x[n]x[n+k_1] \cdots x[n+k_{r-1}]\}$$

- Cumulant of order r again defined over characteristic function
- Can be expressed and estimated via moments
- Fourier transform of ACF is power spectral density
- Fourier transform (2D) of third order cumulant is the *Bispectrum*
- $c_3^x[k_1, k_2] \quad \xrightarrow{\text{DFT}} \quad \mathcal{C}_3^x[\omega_1, \omega_2]$

HOSA – Important Properties

- $x[n]$ Gaussian:

  $$c_r^x(k_1, \ldots, k_{r-1}) = \mathcal{C}_r^x(\omega_1, \ldots, \omega_{r-1}) = 0 \text{ for } r > 2$$

- $x[n]$ i.i.d.:

  $$c_r^x(k_1, \ldots, k_{r-1}) = a \cdot \delta(k_1, \ldots, k_{r-1})$$
  $$\mathcal{C}_r^x(\omega_1, \ldots, \omega_{r-1}) = a$$

- $x[n]$ symmetrically distributed around zero:

  $$c_r^x(k_1, \ldots, k_{r-1}) = \mathcal{C}_r^x(\omega_1, \ldots, \omega_{r-1}) = 0 \text{ for } r = 0, 3, 5, 7, \ldots$$

- $z[n] = x[n] + y[n]$, where $x[n]$, $y[n]$ jointly stationary and statistically independent:

  $$c_r^z(\cdot) = c_r^x(\cdot) + c_r^y(\cdot)$$
  $$\mathcal{C}_r^z(\cdot) = \mathcal{C}_r^x(\cdot) + \mathcal{C}_r^y(\cdot)$$

HOSA – Pros and Cons

Pros

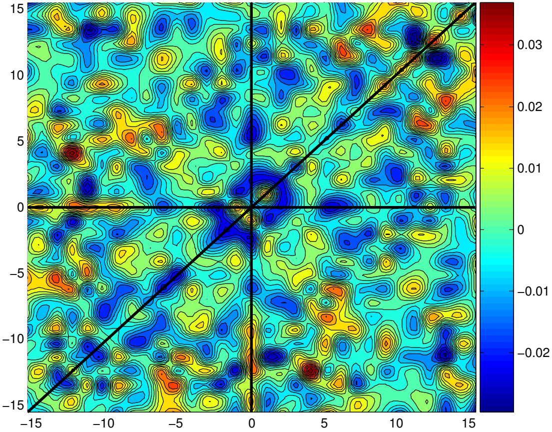
- Analysis of nonlinearities
- Cumulants are additive for independent processes
- Gaussian noise: HOS zero (blind to Gaussian noise)

Cons

- Difficult to estimate from finite length data
- Influence of window
- Once you have them, how do you interpret them?

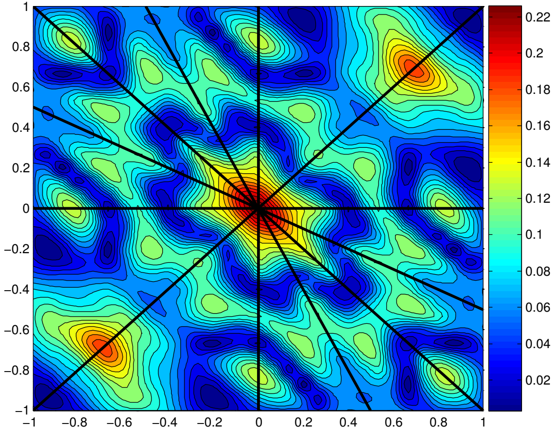
HOSA – Example

- Example for a third-order cumulant



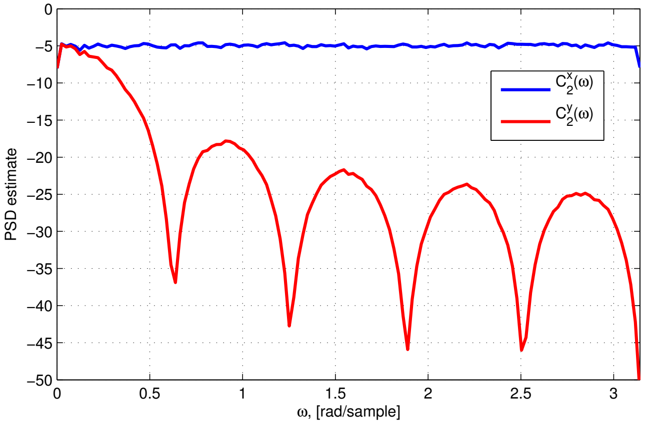
HOSA – Example

- Example for a Bispectrum



HOSA – Example

- Example for a PSD



HOSA – Matlab (1)

- HOSA toolbox, free, included in download-file
- HOSA toolbox manual is a great ressource (Matlab-central)!
- You will need:
  - `cumest.m` used to estimate cumulants
  - `rpiid.m` used to help generating input processes
  - `gabrrao.m` used to calculate window for 2D-FFT
  - `viscumul3.m` and
  - `visbispec3.m` for visualization

HOSA – Matlab (2)

- `cumest.m` for third order cumulant calculates just one slice of the 2D correlation function

```
for k = -MaxLag : MaxLag
    c3(:, k+MaxLag+1) = cumest(#, #, #, #, #, #, k);
end
```

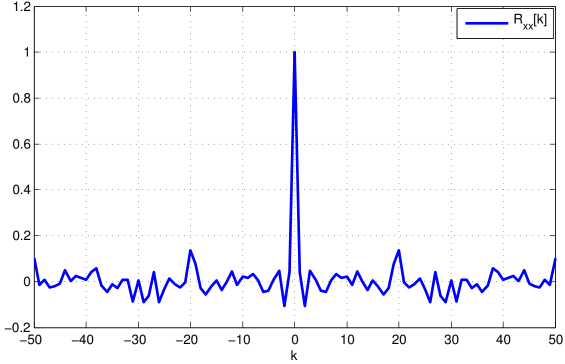
- Bispectrum calculation: use `fftshift(fft2( c3 .* w ))` to have a familiar picture
- Window `w[n]` obtained from `gabrrao.m`, optimal smoothing window, minimum bias in estimation

Higher order statistics and spectral analysis - Problems (1)

- Limited amount of data leads to higher variance of estimators
- Problem even for ACF $\rightarrow$ Grows exponentially with order
- This makes visual interpretation much harder:
  - When is a Bispectrum zero?
  - When is a Bicoherence function flat?

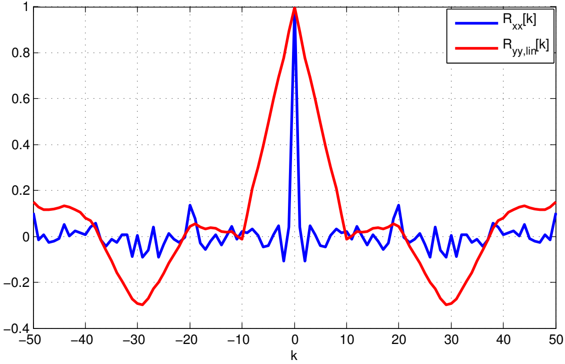
Higher order statistics and spectral analysis - Problems (2)

- Linear system w. $K = 10$, driven by white noise (500 samples)



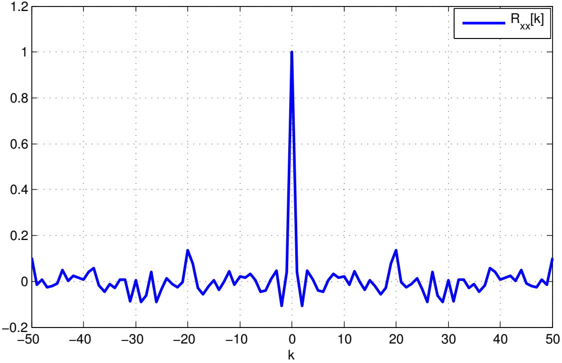
Higher order statistics and spectral analysis - Problems (2)

- Linear system w. $K = 10$, driven by white noise (500 samples)



Higher order statistics and spectral analysis - Problems (2)

- Linear system w. $K = 10$, driven by white noise (500 samples)



Higher order statistics and spectral analysis - Problems (2)

- Linear system w. $K = 10$, driven by white noise (500 samples)