Lesson 4: Non-fading Memory Nonlinearities

Nonlinear Signal Processing – SS 2019

Christian Knoll
Signal Processing and Speech Communication Laboratory
Graz University of Technology

June 13, 2019

Session contents

- Today:
  - Nonlinear dynamical systems
  - Fixed points and local stability
  - Computation of trajectories
  - Discrete-time nonlinear maps
- Next time:
  - Nonlinear maps with chaotic trajectories:
    Bifurcation diagrams and Lyapunov exponents

Nonlinear dynamics – System representation

- A set of equations (nonlinear)
- In continuous time: Differential equations, *flow*

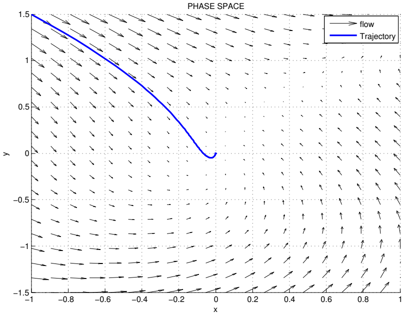
$$\dot{\boldsymbol{x}}(t) = \boldsymbol{f}(\boldsymbol{x}(t))$$

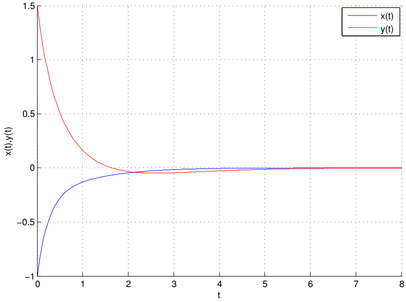
- In discrete time: Difference equations, *map*

$$\boldsymbol{x}[n+1] = \boldsymbol{f}(\boldsymbol{x}[n])$$

Nonlinear dynamics – State space and trajectories

- Variables in the equations span a *phase space*
- Can be x_1 and x_2 for a 2D-eq. or also x and $\dot{x}$ for a 1D eq.
- Flow: Vector field of diff. eq.: e.g. 2D system with x and y
- Derivatives w.r.t. x and y in region of state space



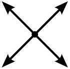
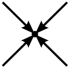
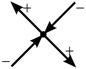


Nonlinear dynamics – Fixed points and local stability

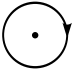
- Fixed point: $\boldsymbol{p}^*$
- At $\boldsymbol{x} = \boldsymbol{p}^*$, we have $\dot{\boldsymbol{x}} = \mathbf{0}$
- Does not mean we are automatically drawn to these points!
- Behaviour of flow/map around $\boldsymbol{p}^*$ defines *local* stability
- Find fixed points as solutions of $\dot{\boldsymbol{x}} = \mathbf{0}$
- Calculate eigenvectors/-values of Jacobian matrix (*local* system linearization) at these points

$$\mathbf{J}(x_0^*, x_1^*, \ldots, x_{N-1}^*) = \left.\begin{pmatrix} \frac{\partial f_1}{\partial x_0} & \frac{\partial f_1}{\partial x_1} & \cdots \\ \frac{\partial f_2}{\partial x_0} & \frac{\partial f_2}{\partial x_1} & \cdots \\ \vdots & & \ddots \end{pmatrix}\right|_{\boldsymbol{p}^*=(x_0^*, x_1^*, \ldots, x_{N-1}^*)}$$

Local Stability I

1. eigenvalues (conjugate) complex and
   - Real part positive: **instable spiral** 
   - Real part negative: **stable spiral** 
2. eigenvalues real and
   - Positive: **repellor** 
   - Negative: **attractor** 
   - Mixed: **saddle** 

Local Stability II

3. Real part zero:
   - analysis of local stability using linearization does not work (linearization behaves differently than NL system)
   - linear behavior: 
4. identical eigenvalues (degenerate node)
   - Jacobian matrix can not be diagonalized
   - Linearization does not capture behavior of the NL
   - stability of linearization similar to NL

- Problem 4.1a as tutorial

Nonlin. dyn. – Attractors

- *A set of points or a subspace in phase space, towards which trajectories converge after transients die out*
- Fixed points are attractors
- Limit cycles are attractors (periodic motion)
- Quasi-periodic motion has an attractor, though same point is never visited twice
- Strange attractors: Is everything the other attractors are not
  - Set of points on which chaotic trajectories move
  - Infinite fine structure, fractal set
  - Extremely sensitive to initial conditions
  - Chaotic trajectories look random, but are not

Nonlin. dyn. – Discrete time – Maps

- In discrete time: Difference equations, *map*

$$\boldsymbol{x}[n+1] = \boldsymbol{f}(\boldsymbol{x}[n]) \quad e.g.: \; x[n+1] = 4rx[n](1-x[n])$$

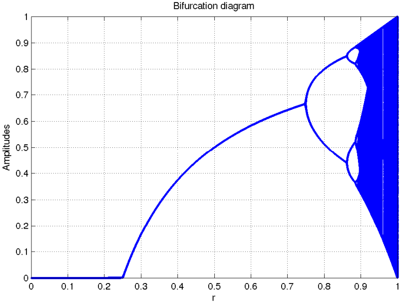
- Fixed points $\boldsymbol{x}^*$ are defined as

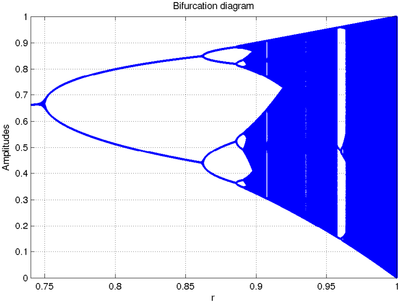
$$\boldsymbol{x}[n+1] = \boldsymbol{x}[n]$$

- Stability and behavior will depend on control parameter r

Nonlinear dynamics – Bifurcation diagram

- Steady-state amplitudes as a function of control parameter
- Transients must have died out!
- E.g.: Logistic map, for $r \approx 0.75$, stable FP splits into two-point-oscillation $\rightarrow$ limit-cycle
- For $r \gtrapprox 0.88$, non-periodic behavior is observed

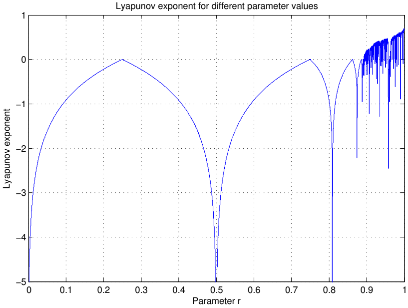




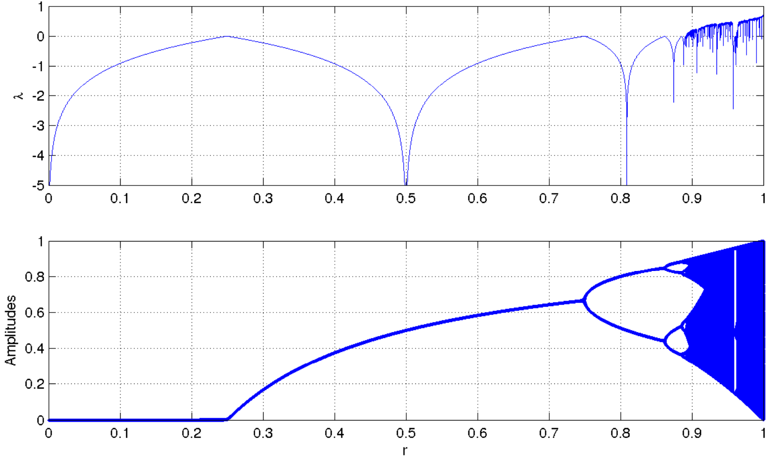
Nonlinear dynamics – Lyapunov exponent

- Measure for sensitivity of trajectories to initial condition
- Stable fixed points: Convergence irrespective of initial point
- Instability can be local, overall amplitude still bounded!
- Lyapunov exponent λ: Like pole radius for linear systems

$$\text{For } x[n+1] = F(x[n]) : \;\; \lambda = \lim_{N\to\infty} \frac{1}{N} \sum_{i=0}^{N-1} \log \left| F'(x[i]) \right|$$



Nonlinear dynamics – Lyapunov exponent and bifurcation



Estimation of Bifurcation diagram and Lyapunov exponent

- Loop over range-of-interest of control parameter r
- Choose an appropriate step size
- For each r: Iterate the map
- Throw away output in transient phase!
- Bifurcation: Record all steady-state amplitudes
- Bifurcation: Plot steady-state amplitudes over range of r
- Lyapunov: Estimate λ by evaluating log of derivative of map at each $x[n]$, then average
- Lyapunov: Plot estimated λ over r