Graz University of Technology
Signal Processing and Speech Communication Laboratory
Nonlinear Signal Processing (course 442.022) Problem set 4
Christian Knoll, christian.knoll@tugraz.at

Non-fading memory systems

Nonlinear dynamics, fixed points and stability

In essence[1], a **flow** is the (non)linear set of equations $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x})$ which form a `phase space` $\mathcal{P}$. In case of difference equations $\boldsymbol{x}[n+1] = \boldsymbol{f}(\boldsymbol{x}[n])$ such a set of equations is called a **map**. As any higher-order differential/difference equation can be transformed into a system of first-order equations, this definition includes higher-order equations as well.

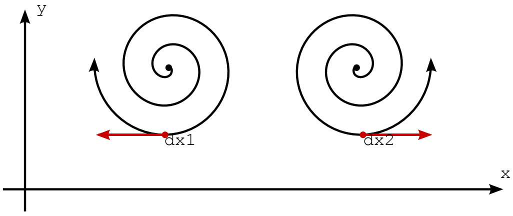
$\mathcal{F}(\dot{\boldsymbol{x}}, \boldsymbol{x}, \dot{\boldsymbol{x}}(0), \boldsymbol{x}(0)) \in \mathcal{P}$ and $\mathcal{F}(\boldsymbol{x}[n], \boldsymbol{x}[0]) \in \mathcal{P}$ are called **trajectories**. Note that the trajectories are (have to be) non-intersecting. Thus, each trajectory corresponds to a unique initial point $(\dot{\boldsymbol{x}}(0), \boldsymbol{x}(0))$ and $\boldsymbol{x}[0]$, resp.

Equilibrium (Fixed-, Stationary-) points

A **fixed point** or **equilibrium point** $\boldsymbol{x}^* = (x_0^*, x_1^*, \dots, x_{N-1}^*)$ is a point in the phase space where the trajectory does not evolve if initialized in this point (the trajectory is a point). Depending on the type of the fixed point (attractor, repellor, ...) the trajectories behave differently in the vicinity of the fixed point. This **local stability** is an important property of a flow or map. A lot of information of the stability of the fixed point can be inferred by linearizing the system around the fixed point, i.e., by the properties of the system's Jacobian matrix evaluated at this fixed point:

$$\mathbf{J}(x_0^*, x_1^*, \dots, x_{N-1}^*) = \left.\begin{pmatrix} \frac{\partial f_1}{\partial x_0} & \frac{\partial f_1}{\partial x_1} & \cdots \\ \frac{\partial f_2}{\partial x_0} & \frac{\partial f_2}{\partial x_1} & \cdots \\ \vdots & & \ddots \end{pmatrix}\right|_{(x_0^*, x_1^*, \dots, x_{N-1}^*)}$$

Depending on the eigenvectors/eigenvalues of the Jacobian matrix, the fixed points are classified as shown on the next page. Each eigenvalue describes the stability in the direction of the eigenvector. The direction of spirals and centers can be obtained by evaluating the flow/map close to the fixed point. The derivative in combination with the stability information gives the direction:

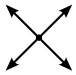
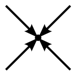
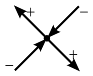


[1] Note that the mathematical definition of "flow" is a lot more generic.

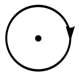
1. eigenvalues (conjugate) complex and

- real part positive: **instable spiral**
- real part negative: **stable spiral**

2. eigenvalues real and

- positive: **repellor** 
- negative: **attractor** 
- mixed: **saddle** 

3. real part of eigenvalues is zero:

- analysis of local stability using linearization does not work (linearization behaves differently than NL system)
- linear behavior: 

4. identical eigenvalues (degenerate node)

- linearization does not capture the full behavior of the NL system
- stability of linearization qualitatively captures stability of the NL system

Problem 4.1

Find and analyze the fixed points of the following equations and plot the flow diagram (vector plot) using `quiver`. Experimentally calculate a trajectory using numerical integration: Use `nde1d.m` for one-dimensional equations and `nde2d.m` for two-dimensional ones. Select an appropriate initial condition.

(a) $\dot{x} = y^2 - x;\ \dot{y} = 2x - y$

(b) $\dot{x} = x - y;\ \dot{y} = x^2 - 4$

Problem 4.2

Perform the same tasks as in Problem 4.1 for the given equations. Modify `nde1d.m` and `nde2d.m` accordingly. Keep in mind that x and y are both functions of time t.

(a) $\dot{x} - e^x + \cos(x) = 0$

(b) $\dot{x} = \sin(y);\ \dot{y} = \cos(x)$

Problem 4.3

A classical system that exhibits chaotic motions for certain parameters is given by the Lorenz equations

$$\begin{aligned}\dot{x} &= \sigma(y - x)\\ \dot{y} &= rx - y - xz\\ \dot{z} &= xy - bz\end{aligned}$$

Plot example trajectories for the initial condition $[x, y, z]^T_{(t=0)} = [0.1,\ 0.1,\ 0.1]^T$. Let $\sigma = 10$, $b = 8/3$ and select r from the set $r \in \{8,\ 166,\ 166.7,\ 28\}$ (use `plot3`). Let the time run from 0 to about 100 seconds and use at least 10^4 points. Also plot x, y and z over time. Describe your observations! You can use `nde3d_lorenz.m` for the Lorenz equations.

Problem 4.4

Consider the Duffing equation given by

$$\ddot{x} + \delta\dot{x} + \alpha x + x^3 = B\cos(wt)$$

Different choices of parameters allow this equation to model a large class of oscillators. You can assume that $\delta > 0$, which represents a damping of the oscillator.

Hint: A higher-order differential equation can be transformed into a system of first-order differential equations using the state vector $\boldsymbol{z}$, where in this case $z_1 = x$ and $z_2 = \dot{x}$.

(a) Assume that $B = 0$. In this case, the equation is also known as the *autonomous* Duffing equation, since there is no driving oscillation present. Find and classify the fixed points (Saddle, Node, ...), analyze their local stability and again plot the vector field using `quiver`. Using example trajectories, illustrate the behaviour of the fixed points.

(b) Assume that $B \neq 0$ and $\omega = 1$, which represents a driving oscillation. Use the following sets of parameters and calculate system trajectories. Plot the time-domain trajectories ($x(t)$ over t).

1. $B = 5,\ \delta = 0.7,\ \alpha = 0$
2. $B = 20,\ \delta = 0.17,\ \alpha = 0$
3. $B = 0.4,\ \delta = 0.25,\ \alpha = -1$

What is going on in these three cases? You can use any representation of the generated $x(t)$ (for a sufficiently long amount of time) you like, but you are not allowed to change the system parameters! You can use the provided function `nde_duffing.m` that uses global variables for the parameters.

Problem 4.5

Perform the following tasks for the logistic map:

(a) Find the fixed points and analyze their stability.

(b) Experimentally verify that if $r \in [0, 0.75]$, for any $x_0 \in [0, 1]$ the output x_n converges to the fixed point. Use `logistic.m` for this task.

(c) Choose $r \in (0.75, 1)$ and iterate the map. What has happened this time and why?

(d) The bifurcation diagram plots the steady-state-amplitudes of a map as a function of the control parameter (see course notes and problem class presentation!). Plot the bifurcation diagram for the logistic map for $r \in [0, \dots, 1]$ and explain the phenomena that you observe.

(e) Compute the Lyapunov exponent for the logistic map for different values of r (i.e. the same range as you did in the previous task). The Lyapunov exponent can be approximated as

$$\lambda = \frac{1}{N} \sum_{i=0}^{N-1} \ln |\dot{f}(x_i)|$$

where N is a number of available trajectory points x_i and $\dot{f}(x_i)$ is a derivative of the map evaluated at the given point. Plot the Lyapunov exponent over r, compare with the bifurcation diagram and describe your observations!

Problem 4.6

Perform similar tasks as in the Problem 4.5 for the following map. Be sure to use a large enough range for both r and x to include the main features of interest. You can easily modify `logistic.m` script to implement the given map.

(a) $x_{n+1} = x_n \mathrm{e}^{-r(1-x_n)}$